The CK Auction

CSCI 1951k/2951z

2020-04-22

These lecture notes offer a high-level summary of two lectures from Professor Tim Roughgarden's Frontiers in Mechanism Design (CS 364B) course, together with references to the relevant sections of his course notes for deeper coverage. The CS 364B lectures are:

- Lecture 2: Unit Demand Bidders and Walrasian Equilibria
- Lecture 3: The Crawford-Knoer Auction

1 *Unit-Demand Valuations*

These two lectures pertain to the **unit-demand** setting, in which bidders are only interested in acquiring one good, but have preferences over which good that might be. The goal of the lectures is to develop an EPIC ascending auction for this setting.

Before attempting to develop an EPIC ascending auction, there is a necessary sanity check—to confirm that there exists a polynomial-time DSIC direct mechanism, assuming unit-demand bidders, because designing the former is at least as hard as designing the latter. Indeed, there exists a polynomial-time algorithm that computes the VCG outcome for the unit-demand setting.

The **winner determination problem**—the task of finding a welfare-maximizing allocation—can be solved in polynomial time, by representing bidders and goods as a bipartite graph and solving for the maximum-weight matching. To compute VCG payments, we remove each bidder from the graph in turn, and re-run the matching algorithm to find that bidder's externality (Lecture 2, pp. 1–3)

2 *Crawford-Knoer (CK) Auction*

The Crawford-Knoer (CK) auction[1] is an ascending auction for the unit-demand setting. The auction is formally described in Lecture 3, pp. 2–3. Recall the EPIC auction design recipe outlined in the EPIC Ascending Auctions lecture:

[1] Vincent Crawford and Elsie Marie Knoer. Job matching with heterogeneous firms and workers. *Econometrica*, 49(2):437–50, 1981

- Design an allocation rule that is welfare maximizing, assuming sincere bidding.
- Show that sincere bidding yields Groves (e.g., VCG) payments.
- Show that inconsistent bidding cannot improve upon sincere bidding for any bidder.

By design, the CK auction, and other similar auctions, such as the KC auction (named for Kelso and Crawford[2]),[3] terminate at a **Walrasian equilibrium** (WE). A WE (Lecture 2, p. 4) is an allocation and pricing such that each bidder ends up with a preferred good at the current prices (WE1); and a good is priced at 0, if it is unallocated (equivalently, if a good's price is positive, then it is allocated; WE2). As prices in ascending auctions can only increase, the CK auction actually terminates at the smallest WE (in terms of prices). Moreover, in the unit-demand setting, the smallest WE is a VCG outcome! Note that this final property is not true more general settings (assuming more general valuations), and will preclude us from obtaining similar incentive guarantees in most settings beyond unit demand.

[2] Alexander Kelso and Vincent Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982

[3] See Lecture 5: Gross Substitutes I.

The following sequence of theorems proves that the CK auction, assuming sincere bidding, terminates near a VCG outcome:

(1) VCG payments lower bound the prices in any WE (Lecture 2, Theorem 3.5).

(2) The VCG outcome is a WE (Lecture 2, Theorem 3.6).

The proof of this claim relies on Lemma 3.7, which states that, in the unit-demand setting, bidder i's VCG payment can be understood as the difference in welfare between injecting into the market an additional copy of the good j that is allocated to i, and the welfare of all bidders (including i) without that additional copy. This difference coincides exactly with i's VCG payment, as injecting into the market an additional copy of the good j that is allocated to i is akin to removing i from the market.

(3) Assuming sincere bidding, the CK auction terminates at an ϵ-WE (Lecture 3, Lemma 3.2). This result is straightforward.

- If a good is not allocated to anyone, it is because no one ever bid on it, in which case its price is zero (WE2).[4]
- If instead, a bidder i wins a good j, it is because j was (one of) i's preferred option(s) when i bid on j. Moreover, j's price did not change since the time at which it was tentatively allocated to i. But the prices of the other goods may have increased. So i could only like j better now than it did at the time when it bid on it. Likewise, for all other bidders k (WE1).

[4] We assume that bidders do not bid if they are indifferent between winning a good and not winning it.

Therefore, by a special case (Lecture 2; Proposition 3.4) of the First Fundamental Theorem of Welfare Economics, which shows that competitive markets allocate resources efficiently,[5] giving support to Adam Smith's "invisible hand" hypothesis, the CK auction terminates with an allocation that is welfare maximizing up to $m\epsilon$.

[5] Technically Pareto-efficiently, which is weaker than welfare maximization, but more broadly applicable, beyond the unit-demand setting.

(4) Assuming sincere bidding, the prices at which the CK auction terminates are upper bounded by the prices of any other WE, up to some additive error δ that depends on ϵ, the number of goods m, and the number of bidders n. Therefore, as the VCG outcome is the smallest WE, the CK auction terminates with VCG payments, up to this δ.

The proof of this claim is a gnarly (Lecture 3, Lemma 3.4). The reverse bound is also necessary (Lecture 3, Lemma 3.5) to prove CK is approximately EPIC: i.e., the prices at which the CK auction terminates are lower bounded by the prices of any other WE, up to some additive error δ that depends on ϵ, the number of goods m, and the number of bidders n.

As sincere bidding in the CK auction is welfare maximizing up to $m\epsilon$, and terminates at VCG payments up to $\delta = \epsilon \min\{m, n\}$, it follows that sincere bidding is an EPNE up to δ among consistent strategies (or up to $m\epsilon$, assuming $m < n$, which is usual).

The final step in the proof that the CK auction is EPIC is to show that no inconsistent strategy is a profitable deviation from sincere bidding up to some additive error 2δ: i.e., no inconsistent strategy yields substantially greater utility than sincere bidding for any bidder (Lecture 3, Theorem 4.2). The proof proceeds by showing that any deviation via an inconsistent strategy can be replicated by a sincere one up to 2δ. Then, since sincere bidding is an EPNE up to δ among (only) consistent strategies, sincere bidding is an EPNE up to $\max\{\delta, 2\delta\} = 2\delta$ among both consistent and inconsistent strategies (or up to $2m\epsilon$, assuming $m < n$, which is usual).

A Proof of the Final Step

Theorem A.1. *The CK auction is EPIC up to* $2\epsilon \min\{m, n\}$.

Proof. Fix some bidder i and valuations $(v_i, \mathbf{v}_{-i})$.

Assume bidder i bids inconsistently and all other bidders bid sincerely. Call the outcome of the CK auction $(\mathbf{x}, \mathbf{q})$. Suppose further that $x_i = j$, meaning i wins good j. (If i does not win anything, then i earns 0 utility, which is the lowest possible utility of any consistent bidding strategy, so inconsistent bidding would not outperform consistent bidding in this case.)

Now assume i had bid sincerely in these auctions, with the following valuation function:

$$v'_{ik} = \begin{cases} \infty & k = j \\ 0 & \text{otherwise} \end{cases}.$$

We claim that the allocation $(\mathbf{x}, \mathbf{q})$ is a Walrasian equilibrium, assuming the bidders' valuations are $(v'_i, \mathbf{v}_{-i})$. Bidder i certainly wins good j, and is utility maximizing at any price (WE1). Further, bidder i does not win any other good at a price greater than 0 (WE2). The rest of the proof of this claim, as it pertains to the other bidders, follows the logic of Lemma 3.2, as all other bidders are bidding sincerely.

The rest of the argument proceeds as follows:

i's utility at outcome $(\mathbf{x}, \mathbf{q})$, after inconsistent bidding by bidder i and sincere bidding by the others

$\leq$ i's utility at allocation $\mathbf{x}$ plus $\epsilon \min\{m, n\}$, assuming Vickrey prices $\mathbf{p}$ and valuations $(v'_i, \mathbf{v}_{-i})$ (1)

$\leq$ i's utility at outcome $(\mathbf{y}, \mathbf{p})$ plus $\epsilon \min\{m, n\}$ in a VCG auction, assuming valuations $(v_i, \mathbf{v}_{-i})$ (2)

$\leq$ i's utility at allocation $\mathbf{y}$ plus $2\epsilon \min\{m, n\}$, assuming sincere bidding in the CK auction and valuations $(v_i, \mathbf{v}_{-i})$ (3)

Step 1 follows because bidder i's allocation is the same in the Walrasian equilibrium assuming valuation v'_i as it is when i bids insincerely in the CK auction. Thus, it suffices to consider prices only, when comparing i's utility between inconsistent and sincere bidding; and, among all prices that support an efficient allocation at Walrasian equilibrium (i.e., any price between the highest and the second-highest values), Vickrey prices are the smallest up to $\epsilon \min\{m, n\}$ (Lemma 3.5).

Step 2 follows from the fact that VCG is DSIC, so it cannot benefit bidder i to bid according to v'_i rather than v_i in a VCG auction.

Step 3 follows because, assuming sincere biding, the CK auction converges to a Walrasian equilibrium (Lemma 3.2), and in particular the smallest prices (i.e. the Vickrey prices) that support an efficient allocation, in this case $\mathbf{y}$,[6] up to $\epsilon \min\{m, n\}$ (Lemma 3.4). □

[6] If necessary, ties can be broken to match allocation $\mathbf{y}$.

References

[1] Vincent Crawford and Elsie Marie Knoer. Job matching with heterogeneous firms and workers. *Econometrica*, 49(2):437–50, 1981.

[2] Alexander Kelso and Vincent Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.