Order Statistics and Revenue Equivalence

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We derive the first and second order statistics for the uniform distribution on $[0,1]$. We use these results to prove that the expected revenue of the first-price auction is equal to that of the second-price auction.

1 *Order Statistics*

Definition 1.1. The k**th-order statistic**, denoted $X_{(k)}$, is the the kth-largest realization, among n, of a random variable X.

In particular, the first order statistic is the maximum of n draws, the second order statistic is the second highest of n draws, and the nth-order statistic is the minimum of n draws.

Order statistics, for reasons that should be intuitively clear, are useful in analyzing the outcome of first- and second-price auctions.

2 *First Order Statistic*

We are interested in calculating the expected value of $X_{(1)}$, the first order statistic, when sampling i.i.d. from a uniform distribution, call it U, on $[0,1]$. That is,

$$\mathbb{E}\left[X_{(1)}\right] = \int_0^1 x f_{X_{(1)}}(x)\ dx.$$

We will proceed by computing the CDF $F_{X_{(1)}}$, which is easy to compute, and then taking derivatives to arrive at the PDF, $f_{X_{(1)}}$.[1]

Observe that the CDF at some value $x \in [0,1]$ is the probability that all n draws are less than x: i.e.,

$$\begin{aligned} F_{X_{(1)}}(x) &= \Pr(X_{(1)} \leq x) \\ &= \prod_n U(x) \\ &= x^n. \end{aligned}$$

Now

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{d}{dx} F_{X_{(1)}}(x) \\ &= \frac{d}{dx} x^n \\ &= n x^{n-1}. \end{aligned}$$

Therefore,

$$\mathbb{E}\left[X_{(1)}\right] = \int_0^1 x f_{X_{(1)}}(x)\ dx = \int_0^1 n x^n\ dx = \frac{n}{n+1}.$$

[1] The CDF is defined as follows:

$$F_X(x) = \int_{-\infty}^x f_X(t)\ dt.$$

By the Fundamental Thm of Calculus,

$$f_X(x) = \frac{d}{dx} F_X(x).$$

In particular,

$$f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x).$$

2.1 *Second Order Statistic*

We follow the same steps to compute the second order statistic (using fewer words).

CDF:

$$\Pr(X_{(2)} \leq x) = x^n + nx^{n-1}(1-x)$$

In words, all the samples can be less than x, which happens with probability x^n, *or* only $n-1$ of the samples can be less than x, which can happen in n different ways, each with probability $x^{n-1}(1-x)$.

PDF:

$$f_{X_{(2)}}(x) = nx^{n-1} + n(n-1)x^{n-2}(1-x) - nx^{n-1} = n(n-1)x^{n-2}(1-x)$$

Expected value of the second order statistic:

$$\begin{aligned}
\mathbb{E}\left[X_{(2)}\right] &= \int_0^1 x f_{X_{(2)}}(x)dx \\
&= n(n-1)\int_0^1 \left(x^{n-1} - x^n\right) dx \\
&= n(n-1)\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= n(n-1)\frac{1}{n(n+1)} \\
&= \frac{n-1}{n+1}.
\end{aligned}$$

3 *Revenue Equivalence*

Theorem 3.1. *If bidder's values are uniform i.i.d., then the expected revenue of the first-price auction is equal to that of the second-price auction, assuming bidders behave according to their respective equilibrium strategies.*

Proof. The support of the uniform distribution does not matter; we choose $[0,1]$ for convenience. Let R_1 and R_2 denote the expected revenue of the first- and second-price auctions, respectively.

In the second-price auction, the bidder with the highest value wins, paying the second-highest value. Therefore, the expected revenue is equal to the expected value of the second order statistic: i.e.,

$$R_2 = \frac{n-1}{n+1}.$$

In a first-price auction, the expected revenue is equal to the expected highest bid. Recall that the equilibrium bid function $b_i = \left(\frac{n-1}{n}\right) v_i$. As this function is monotonically non-decreasing, the highest bidder also has the highest valuation, call it $v_{\max}$. In other words,

$$R_1 = \mathbb{E}\left[\left(\frac{n-1}{n}\right) v_{\max}\right] = \left(\frac{n-1}{n}\right) \mathbb{E}\left[v_{\max}\right]$$

But $\mathbb{E}[v_{\max}]$ is the expected value of maximum of n i.i.d. draws from random variable uniformly distributed on $[0,1]$ (i.e., the expected value of the first order statistic), which is $\frac{n}{n+1}$. Thus,

$$R_1 = \left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right) = \frac{n-1}{n+1}.$$

Therefore, $R_1 = R_2$. □

A kth Order Statistic

Beta Function The Beta function $\mathrm{B}(x,y)$ is defined by the following integral:

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dy.$$

When x and y are positive integers, this function simplifies as follows:

$$\mathrm{B}(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

We will use the Beta function in (the very last step of) our derivation of the expected value of the kth order statistic.

We begin by computing the probability the kth order statistic lies in some small interval $[x, x+\Delta x] \subset [0,1]$. When the draws from X are i.i.d.,

$$\mathrm{P}(X_{(k)} \in [x, x+\Delta x]) = n\binom{n-1}{k-1}\,\mathrm{P}(X < x)^{n-k}\,\mathrm{P}(X \in [x, x+\Delta x])\,\mathrm{P}(X > x+\Delta x)^{k-1} + \mathrm{O}(\Delta x^2)$$

The middle three probabilities are, respectively, the chance of:

- exactly $n-k$ values less than x,
- exactly one value between x and $x+\Delta x$, and
- exactly $k-1$ values greater than $x+\Delta x$.

This gives the probability of one specific arrangement of this form, so we multiply by the number of possible arrangements. There are n possible agents who could have a value between x and $x+\Delta$, after which there are $\binom{n-1}{k-1}$ possible groups of agents who could have values greater than x, after which the remaining $n-k$ agents are fixed. **N.B.** There is also a chance that multiple values fall between x and $x+\Delta x$. As each such probability will contain a Δx^i term with $i \geq 2$, we include the term $\mathrm{O}(\Delta x^2)$.

The assumption that X is uniformly distributed on $[0,1]$ yields the following further simplification:

$$\mathrm{P}(X_{(k)} \in [x, x+\Delta x]) = n\binom{n-1}{k-1}\,x^{n-k}\,\Delta x\,(1-x-\Delta x)^{k-1} + \mathrm{O}(\Delta x^2)$$

Letting $x_{i+1} = x_i + \Delta x$, we can express the expectation of interest in discretized space as follows:

$$\sum_{i=1}^{m} x_i \, \mathrm{P}(\mathrm{X}_{(k)} \in [x_i, x_{i+1}])$$

To calculate the corresponding continuous expectation, we take the limit as $m \to \infty$, so that the Δx terms become arbitrarily small:

$$\begin{aligned}
\mathbb{E}\left[X_{(k)}\right] &= \lim_{m\to\infty} \sum_{i=1}^{m} x_i \, \mathrm{P}(X_{(k)} \in [x_i, x_i + \Delta x]) \\
&= n\binom{n-1}{k-1}\left(\lim_{m\to\infty} \sum_{i=1}^{m} x_i^{n-k+1} \, \Delta x \, (1 - x_i - \Delta x)^{k-1} + \mathrm{O}(\Delta x^2)\right) \\
&= n\binom{n-1}{k-1} \int_0^1 x^{n-k+1}(1-x)^{k-1} dx \\
&= n\binom{n-1}{k-1} \mathrm{B}(n-k+2, k) \\
&= \left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{(n-k+1)!(k-1)!}{(n+1)!}\right) \\
&= \frac{n-k+1}{n+1}.
\end{aligned}$$