Primal-Dual Auctions

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1 *An Ascending Auction for Unit-Demand Valuations*

Last week, we saw the CK auction for unit-demand valuations and analyzed its properties. This week, we see another ascending auction for the unit-demand model from an entirely different perspective. As we recollect, in the unit-demand model, we have a set N of n bidders and a set G of m heterogenous goods such that each bidder values any subset of goods $S \subseteq G$ at the value of the single most valuable good in it, i.e.,

$$v_i(S) = \max_{j \in S} v_{ij}$$

We assume that all valuations are integers. We would like to design an EPIC ascending auction for this setting. From our recipe, it is sufficient to design an auction that results in the VCG outcome for sincere bidding and show that inconsistent bidding cannot help any bidder do better than consistent bidding. First, we discuss an ascending auction design due to Demange, Gale and Sotomayor [1] that terminates with the VCG outcome.

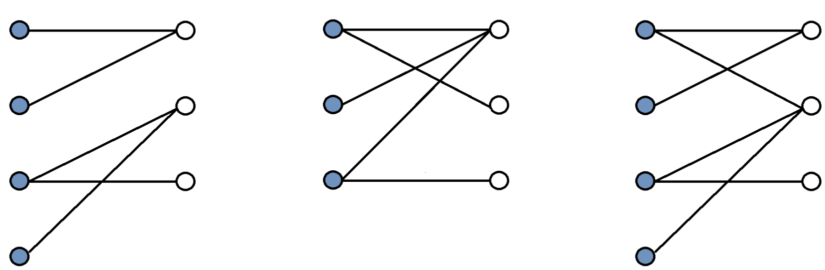
[1] Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94(4):863–872, 1986

An Aside. Before we begin we recollect Hall's marriage theorem for matching in bipartite graphs. Given a bipartite graph with vertices $X + Y$ and edges E there exists an X-saturating matching[2] in this graph if and only if the following condition holds for all $S \subseteq X$:

[2] a matching that matches all the vertices in X to some vertex in Y.

$$|S| \leq |N(S)|$$

where $N(S) = \{v \in Y \mid \exists u \in S s.t. (u, v) \in E\}$ is the set of neighbors of S in Y. The condition requires that *every* subset of X have enough neighbors in Y for a matching to be possible. A set S for which this condition does not hold is called a *Hall violator*.



In the above examples, the first and the last graph contain Hall violators. For any bipartite graph $(X + Y, E)$, there is an efficient algorithm that will either return an X-saturating matching or return a subset that is a *Hall violator*. In fact, the algorithm returns a minimal Hall violator i.e. a subset S that violates the Hall marriage condition such that no $X \subset S$ is a Hall violator. Which of the above Hall violators are minimal? Let the algorithm be denoted MIN-HALL-VIOLATOR for future reference.

In the unit-demand setting an allocation corresponds to a matching between goods and bidders. If we consider $X = N$ and $Y = G$ and the edges

$$E = \{(u, v) \mid \text{bidder } u \text{ demands good } v \text{ at the current prices}\},$$

we can define an *over-demanded* set of goods. An over-demanded set of goods is a set of goods $S \subseteq G$ such that the number of bidders that demand *only* goods in S is greater than the size of S. In other words, the set of bidders $T \subseteq X$ bidding on these goods have less neighbors in Y than the size of T. This is exactly the condition for a Hall violator in the set of bidders. From Hall's theorem, we know that there exists an X-saturating matching in a bipartite graph if and only if there are no Hall violators in the set X. In the auction setting, this implies that we cannot have a matching in the demand graph if there exists an over-demanded set of goods at the current prices. An over-demanded set of goods would imply that there is no way to make the bidders bidding on these goods happy. This violates the condition WE1 for a Walrasian equilibrium. In fact, there exists a Walrasian equilibrium if and only if there are no over-demanded sets of goods at the current prices [3].

[3] David Gale. *The theory of linear economic models*. University of Chicago press, 1989

This gives us some intuition to develop the auction design, and we are ready to describe the auction for unit-demand valuations as follows:

- Initialize the prices of all the goods to zero, $p_j = 0, \forall j$.
- At every round r:
  - ask every bidder to report *all* the goods they are willing to buy at the current prices $\mathbf{p}$. This set is called the demand set, denoted as $D_i(\mathbf{p})$ for bidder i;
  - given the demand graph G_D between goods and bidders, run MIN-HALL-VIOLATOR(G_D). If the algorithm returns a matching terminate with that allocation and current prices;
  - if such a matching is not possible, the algorithm identifies a set of bidders that demand a minimal set of over-demanded goods;
  - the prices on this set of goods are increased by $\epsilon = 1$[4] and the auction continues to the next round.

[4] this choice is reasonable because we assume valuations are integral.

1.1 An Example Auction

As an example, consider the following auction with goods $G = \{g_1, g_2, g_3\}$ and bidders $N = \{b_1, b_2, b_3, b_4\}$. Let the valuations of the bidders be as follows:

$$
\begin{array}{llll}
b_1 \to & v_{11} = 1; & v_{12} = 2; & v_{13} = 3; \\
b_2 \to & v_{21} = 3; & v_{22} = 2; & v_{23} = 1; \\
b_3 \to & v_{31} = 2; & v_{32} = 1; & v_{33} = 3; \\
b_4 \to & v_{41} = 1; & v_{42} = 2; & v_{43} = 5;
\end{array}
$$

The auction execution is depicted in the table below. At round r, D_i is the demand set of bidder b_i at the prices $\mathbf{p}$ and $p_{j,r}$ is the price of good g_j. O is the minimal over-demanded set of goods whose prices will be increased in round $r+1$. Then we can see the auction proceeds as follows:

Round (r)	$p_{1,r}$	$p_{2,r}$	$p_{3,r}$	D_1	D_2	D_3	D_4	O
1	0	0	0	$\{3\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
2	0	0	1	$\{2,3\}$	$\{1\}$	$\{1,3\}$	$\{3\}$	$\{1,3\}$
3	1	0	2	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{3\}$	$\{1,2,3\}$
4	2	1	3	$\{2\}$	$\{1,2\}$	$\{\}$	$\{3\}$	$\{\}$

The auction terminates at round $r = 4$ with the outcome of g_1 to b_2 at price 2, g_2 to b_1 at price 1, and g_3 to b_4 at price 3. Bidder 3 is priced out of the market at round 4. By its ascending nature, this auction is guaranteed to terminate. Since the prices are always rising, the demand for the goods will drop off eventually. When the auction terminates there will be no over-demanded goods. We already know that (1) no over-demand is a sufficient condition for the existence of Walrasian equilibrium at current prices and (2) there exists a matching in the demand graph at final prices that allocates every bidder. Then we can see that the auction terminates at a Walrasian equilibrium.

Additionally, the auction terminates at the smallest Walrasian equilibrium, which (we have seen) corresponds to the VCG outcome in the unit-demand setting. We state and prove the following theorem from Demange, Gale and Sotomayor [5].

[5] Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94(4):863–872, 1986

Theorem 1.1. *Let* $\mathbf{p}$ *be the price vector obtained from the auction under sincere bidding and* $\mathbf{q}$ *be any other competitive price vector. Then* $\mathbf{p} \leq \mathbf{q}$.

Proof. For contradiction, let us assume that $\mathbf{p}$ is not the smallest Walrasian price vector. Let $\mathbf{q}$ be the smallest Walrasian price vector instead. The prices $\mathbf{p}$ are increased in every round $1 \leq r \leq n$ of the auction. Let the price vector at round r be denoted as $\mathbf{p}_r$. Then initially $\mathbf{p}_1 = \vec{0}$ and therefore $\mathbf{p}_1 \leq \mathbf{q}$. But at termination, $\mathbf{p}_n \geq \mathbf{q}$. This implies there is a round $t, 1 < t < n$ at which $\mathbf{p}_t \leq \mathbf{q}$ but $\mathbf{p}_{t+1} \geq \mathbf{q}$. Let us examine this round of the auction. Let S be the (minimal) over-demanded set of goods in round t whose prices are increased in round $t+1$. Let T be the set of bidders who demand only goods in S. By the definition of over-demand:

$$|T| > |S| \tag{1}$$

Our strategy will be to show that there exists a proper subset of S that is over-demanded. This cannot be possible according to the rules of the auction thereby creating a contradiction. We have the following observations:

- Let S_1 be the set of goods j such that $p_{j,t+1} > q_j$. But during round t all goods j had prices $p_{j,t} \leq q_j$ and the prices increased by 1. Then it has to be that $p_{j,t} = q_j$ for all the goods in S_1. This means that all the goods in S_1 were priced at Walrasian prices in round t.

- We assume for now that $S - S_1$ is non-empty. Let T_1 be the bidders who demand at least one good in set S_1 under prices $\mathbf{p}_t$. We show that their demanded goods under prices $\mathbf{q}$ will all be in S_1. Consider one bidder $i \in T_1$ and let $\alpha \in S_1$ be a good they demand in S_1.

  1. For any $\beta \in S - S_1$, the bidder i likes α at least as much as β under $\mathbf{p}_t$. However, since $\alpha \in S_1$, we know $p_{\alpha,t} = q_\alpha$. Also, since $\beta \in S$ the price of β increased after round t. But $\beta \notin S_1$ so the new price is still at most the price of β in $\mathbf{q}$. Then $p_{\beta,t} < p_{\beta,t+1} \leq q_\beta$. If α maximizes bidder i's utility even when β is cheaper than in $\mathbf{q}$, bidder i will also prefer α to β under prices $\mathbf{q}$.
  2. For any $\beta \notin S$, the bidder i prefers α more than β at $\mathbf{p}_t$ even though $p_{\alpha,t} = q_\alpha$ and β is cheaper since $p_{\beta,t} \leq q_\beta$. Similarly bidder i will still prefer α to β under prices $\mathbf{q}$.

  Then for all $i \in T_1$ their demand sets under prices $\mathbf{q}$ will lie inside S_1.

- However since $\mathbf{q}$ is an equilibrium there cannot be over-demand under $\mathbf{q}$. Therefore the set of bidders T_1 must demand enough goods under $\mathbf{q}$:
$$|T_1| \leq \left| \bigcup_{i \in T_1} D_i(\mathbf{q}) \right| \leq |S_1| \tag{2}$$

- From equations (1, 2) we see that $|T - T_1| > |S - S_1|$. We assumed that $S - S_1$ is non-empty, and we know that all the bidders in

$T - T_1$ have no demanded goods in S_1 by the definition of T_1. It has to be that the bidders $T - T_1$ want only goods in $S - S_1$. Then $S - S_1$ is a proper subset of S that is over-demanded and we have the contradiction we were looking for.

- To complete the proof, we show that $S - S_1$ cannot be empty. For contradiction, assume it is empty i.e. all the goods in S are also in S_1. From our earlier observation about S_1, all the goods in S now have prices $p_{t,j} = q_j$. The bidders T only demand goods in S from the over-demand condition. But all the goods β outside S have prices $p_\beta \le q_\beta$. If the bidders in T prefer goods in S to all such β under $\mathbf{p}_t$ they will continue to demand them under the prices $\mathbf{q}$. But $\mathbf{q}$ is an equilibrium price vector and no set of goods can be over-demanded. Therefore S cannot be over-demanded under $\mathbf{q}$:

$$|T| \le \left| \bigcup_{i \in T} D_i(\mathbf{q}) \right| \le |S|$$

  but this violates the over-demand condition on S at round t since the same bidders are also bidding on S at prices $\mathbf{p}_t$. With this, our proof is complete.

$\square$

Now we know that the auction terminates with the minimum Walrasian price vector $\mathbf{p}$ and an allocation under these prices. From the last lecture, we know that the VCG payments are the same as the minimum Walrasian prices. From the first welfare theorem, we know that all allocations supported by Walrasian prices are welfare-maximizing. Given that the outcome is welfare-maximizing and the payments are the VCG payments, we know that the auction outcome is the same as the VCG outcome.

In the context of our recipe for EPIC auctions, we now only have to show that inconsistent bidding does not do better than consistent bidding. We note that a similar argument to last week's for the CK auction works in this setting. In today's lecture we will study the connection between linear programs and ascending auctions, and derive the above auction from a completely different viewpoint.

2 *The Primal-Dual Method for Auctions*

In the unit-demand setting it is possible to express the welfare-maximizing allocation as the objective of an integer linear program as follows:

$$\begin{array}{llrcll}
\max_x & \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}x_{ij} & & & & \\
\text{s.t.} & & \sum_{i=1}^{n} x_{ij} & \leq & 1 & \forall j \in G \\
 & & \sum_{j=1}^{m} x_{ij} & \leq & 1 & \forall i \in N \\
 & & x_{ij} & \in & \{0,1\} & \forall i \in N, j \in G
\end{array}$$

This program maximizes the welfare of the allocated bidders, subject to some constraints. The first constraint says that no good can be allocated to more than one bidder. Similarly the second constraint says that no bidder can be allocated more than one good. The last constraint forces the allocation to be integral i.e. the goods we are dealing with are indivisible. We can relax this integer program to the linear program because we know that this particular linear program has integer optimal solutions [6].

[6] Lloyd S Shapley and Martin Shubik. The assignment game i: The core. *International Journal of game theory*, 1(1):111–130, 1971

$$\begin{array}{llrcll}
\max_x & \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}x_{ij} & & & & \\
\text{s.t.} & & \sum_{i=1}^{n} x_{ij} & \leq & 1 & \forall j \in G \\
 & & \sum_{j=1}^{m} x_{ij} & \leq & 1 & \forall i \in N \\
 & & x_{ij} & \geq & 0 & \forall i \in N, j \in G
\end{array} \tag{3}$$

This linear program is the *primal* (P) that we will work with today. Following the usual method, we derive the dual (D) of this program to be the following:

$$\begin{array}{llrcll}
\min_{p,u} & \sum_{i=1}^{n} u_i + \sum_{j=1}^{m} p_j & & & & \\
\text{s.t.} & & u_i + p_j & \geq & v_{ij} & \forall i \in N, j \in G \\
 & & u_i & \geq & 0 & \forall i \in N \\
 & & p_j & \geq & 0 & \forall j \in G
\end{array} \tag{4}$$

The dual variables in this program can conveniently be interpreted as the prices of the goods p_j and the utilities of each bidder u_i. We denote the vectors of prices and utilities as $\mathbf{p}, \mathbf{u}$ respectively. Together the primal and the dual programs define the outcome of the auction, which is a Walrasian outcome as we see in the following theorem [7].

[7] Lloyd S Shapley and Martin Shubik. The assignment game i: The core. *International Journal of game theory*, 1(1):111–130, 1971

Theorem 2.1. $\mathbf{p}$ *is a Walrasian price vector if and only if it participates in an optimal solution* $(\mathbf{p}, \mathbf{u})$ *to the dual.*

Proof Sketch. We note that the complementary slackness conditions for the above primal and dual are equivalent to the conditions for

Walrasian Equilibria. The complementary slackness conditions for the programs above are as follows:

1. $x_{ij} > 0 \Rightarrow u_i = v_{ij} - p_j$ and $u_i > v_{ij} - p_j \Rightarrow x_{ij} = 0$.
2. $\sum_{i=1}^{n} x_{ij} < 1 \Rightarrow p_j = 0$.
3. $\sum_{j=1}^{m} x_{ij} < 1 \Rightarrow u_i = 0$.

Let $X = \{x_{ij}\}$ be the optimal primal solution and $(\mathbf{p}, \mathbf{u})$ be the optimal dual solution[8]. X is a matching of bidders to goods such that: (1) every bidder who is allocated gets one of their favorite goods; (2) if a good is not allocated its price is zero; (3) if a bidder is not allocated their utility is zero. These are exactly the conditions for Walrasian equilibrium. In the other direction if $(\mathbf{p}, X)$ is a Walrasian equilibrium with utilities $\mathbf{u}$ such that $u_i = \max_j(v_{ij} - p_j)$ then $X, (\mathbf{p}, \mathbf{u})$ must satify the conditions WE1, WE2 which are the same as the complementary slackness conditions. Then X is a primal feasible matching and $(\mathbf{p}, \mathbf{u})$ are a dual feasible solution such that they satisfy the complementary slackness conditions. Therefore they must be optimal.

[8] both solutions are integral as we have seen.

In the next section, we use the primal-dual algorithm to solve the linear program (P) and derive an ascending auction in the process.

3 A Primal-Dual Algorithm

Given the programs (P) and (D), we know by strong duality that at the optimal value for both programs,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}x_{ij}^* = \sum_{i=1}^{n} u_i^* + \sum_{j=1}^{m} p_j^*.$$

On the left of the equation we have the welfare of the allocation, and on the right we have the sum of the prices and the utilities of the individual bidders. We also know that the optimal solution to the dual gives us Walrasian prices from Theorem 2.1.

At a high level, the primal-dual algorithm arrives at the optimal value through a series of adjustments of the objective value. The steps are (roughly) as follows:

1. Find a feasible solution to the dual.
2. Using this solution, write a 'restricted' primal program (RP) that expresses the objective of finding the corresponding primal-feasible solution that obeys complementary slackness.

3. Compute the dual of the restricted primal (DRP). This program can often be solved combinatorially and has a form similar to the original dual.

4. Use the solution to (DRP) to 'improve' the solution to the original dual.

3.1 The Restricted Primal

We begin by calculating a feasible solution to the dual (D). For every good j, set the price $p_j = 0$. For every bidder i, set $u_i = \max_j(v_{ij} - p_j)$. This is a dual-feasible solution. For each i, the set of goods that maximize (positive) u_i at the current prices are the demand set, $D_i(\mathbf{p})$. Then we define the set of edges E of the (bipartite) demand graph as follows:

$$E = \{(i,j) \mid j \in D_i(\mathbf{p}), u_i > 0\}$$

Similarly let the set of bidders that demand good j be denoted as $B_j(\mathbf{p})$. If there exists a matching in the demand graph that allocates goods to all the bidders currently demanding goods, we are done. If there does not exist such a matching, then we formulate the restricted primal program. Let $X = B^e$ denote the set of bidders and $Y = G^e$ the set of goods in the demand graph. Let G_0^e be the set of goods that have prices 0 in the dual solution we started with:

$$\begin{array}{llcll}
\min\limits_{x,y,z} & \sum\limits_{i \in B^e} y_i + \sum\limits_{j \notin G_0^e} z_j & & & \\
\text{s.t.} & \sum\limits_{i \in B_j(\mathbf{p})} x_{ij} & \leq & 1 & \forall j \in G_0^e \\
& \sum\limits_{i \in B_j(\mathbf{p})} x_{ij} + z_j & = & 1 & \forall j \notin G_0^e \\
& \sum\limits_{j \in D_i(\mathbf{p})} x_{ij} + y_i & = & 1 & \forall i \in B^e \\
& x_{ij} & \geq & 0 & \forall (i,j) \in E \\
& y_i & \geq & 0 & \forall i \in B^e \\
& z_j & \geq & 0 & \forall j \notin G_0^e
\end{array}$$

where y, z are newly defined 'slack' variables. We note that we start the prices at 0, so all the goods that are part of G^e in the first iteration have price 0 and therefore $G^e = G_0^e$. The first constraint gives some slack on the primal constraint for goods if the prices are 0. If the prices are non-zero, the second constraint enforces (relaxed) complementary slackness on goods. The slack variables z_j are therefore only defined for $j \notin G_0^e$. For bidders in B^e, the value of $u_i > 0$ because of how E is defined. When we begin all the bidders are in B^e[9]. The second constraint enforces (relaxed) complementary slackness on

[9] we can ignore any bidder who is not part of B^e at prices 0, because if they don't demand any goods at price 0 they do not have any incentive to participate in the auction.

these bidders. At the optimal dual solution, there will exist a primal-feasible solution that obeys the complementary slackness conditions. This implies that all the slack variables can take value 0 and so the optimal value of the restricted primal objective will be 0. For any sub-optimal dual solution, there will not exist such a primal solution. Then the slack variables will have to be positive and the optimal objective value of the restricted primal will also have a positive value.

3.2 *Improving the Dual Solution*

By strong duality, when the optimal objective value of the restricted primal is positive the optimal objective value of the dual of the restricted primal will also be positive. Then we compute the dual of the restricted primal as follows:

$$\begin{array}{llrcll}
\max\limits_{u',p'} & -\sum\limits_{i\in B^e} u'_i - \sum\limits_{j\in G^e} p'_j & & & & \\
\text{s.t.} & & u'_i + p'_j & \geq & 0 & \forall (i,j) \in E \\
& & u'_i + 1 & \geq & 0 & \forall i \in B^e \\
& & p'_j + 1 & \geq & 0 & \forall j \notin G^e_0 \\
& & p'_j & \geq & 0 & \forall j \in G^e_0
\end{array}$$

In short, if the dual feasible solution we started with was not optimal, then the above program will have a positive optimal value. Then there exist some values for p', u' such that $\sum_{i\in B^e} u'_i + \sum_{j\in G^e} p'_j$ will be negative. How do we find such values? To be feasible, the p' variables have to be at least 0 for some goods (and -1 for others depending on their current prices), the u' have to be at least (-1) and they must sum to at least 0 for every edge in E. Additionally, the number of bidders must somehow outweigh the number of goods for $\sum_{i\in B^e} u'_i + \sum_{j\in G^e} p'_j$ to have a negative value. Does this sound familiar?

If we can find a minimal overdemanded set of goods we can construct such a solution to the dual of the restricted primal. Let the overdemanded set of goods be O and the bidders bidding only on goods in O be O_b. Construct a solution as follows:

1. Set the price $p'_k = 0$ for every good $k \notin O$.
2. Set the utility $u'_l = 0$ for every bidder $l \notin O_b$.
3. For every good j in O, set $p'_j = 1$.
4. For every bidder i in O_b set $u'_i = (-1)$.

This is a feasible solution with $\sum_{i\in B^e} u'_i + \sum_{j\in G^e} p'_j < 0$. How do we find such a set? Run MIN-HALL-VIOLATOR on the demand graph!

Now we have a solution to reduce the objective value of the dual. Increase the p_j for all the goods j in the set O by $\epsilon = 1$ and decrease

u_i by 1 for all the bidders in O_b. Since the bidders demanded only items in O, their maximum utility decreases by 1 because all the goods they wanted are now more expensive. Adding a negative value to the dual objective reduces its value, bringing it closer to the optimal point. This is exactly the price rise step in the Demange auction. We then start with a new dual-feasible solution, formulate the restricted primal corresponding to it, and repeat till the algorithm terminates.

3.3 *Termination*

The algorithm terminates when the dual of the restricted primal has value 0 i.e. there are no over-demanded subsets of goods, which is sufficient for equilibrium. Then MIN-HALL-VIOLATOR will return a matching in the demand graph. To complete the picture, we argue that the complementary slackness conditions are maintained throughout the execution of the primal-dual algorithm.

1. $x_{ij} > 0 \Rightarrow u_i = v_{ij} - p_j$ and $u_i > v_{ij} - p_j \Rightarrow x_{ij} = 0$. The utility u_i is always maintained at the maximum possible value. If a bidder is allocated, their edge was in the demand graph, and therefore they receive a good that maximizes their utility.

2. $\sum_{j=1}^{m} x_{ij} < 1 \Rightarrow u_i = 0$. If a bidder has positive utility, they will be part of the demand graph and therefore the final matching. Then the only bidders that do not get allocated are those who have utility 0 at the current prices.

3. $\sum_{i=1}^{n} x_{ij} < 1 \Rightarrow p_j = 0$. Lastly, if the price of a good increases once it is then part of some over-demanded set of goods O. This implies it is in the demand set of at least one bidder who demands only goods in O. The good remains in the demand set of this bidder till the bidder is 'priced out' of the competition for that good (utility drops to zero). At that point, this implies that some other bidder is demanding the good[10]. Then if the price of a good increases from zero, it continues to be in at least one demand set until the final matching allocates it to a bidder. Therefore if a good is not allocated to any bidder, it never was in the demand graph and therefore is still priced at 0 when the auction ends.

[10] assuming not everyone is priced out in the same round.

Then when the algorithm terminates, it will terminate with a solution to the dual that cannot be improved any further since there are no overdemanded sets of goods. This is then a Walrasian equilibrium. From Theorem 2.1 every solution to the dual is a Walrasian price vector. Additionally, at every iteration we increase the prices of a minimal over-demanded set of goods. This process leads to the smallest Walrasian prices from Theorem 1.1. From our earlier reasoning for the auction, this algorithm also terminates with the VCG outcome.

3.4 *The Example, Revisited*

We now revisit our earlier auction with 3 goods and 4 bidders with this new algorithm. Recall the valuations as follows:

$$\begin{array}{lll} v_{11} = 1; & v_{12} = 2; & v_{13} = 3; \\ v_{21} = 3; & v_{22} = 2; & v_{23} = 1; \\ v_{31} = 2; & v_{32} = 1; & v_{33} = 3; \\ v_{41} = 1; & v_{42} = 2; & v_{43} = 5; \end{array}$$

- We begin with a dual feasible solution. Let all the prices of the goods $p_j = 0$. Then for each bidder i the current utility $u_i = \max_j(v_{ij} - p_j)$. We have $u_{b_1} = 3, u_{b_2} = 3, u_{b_3} = 3, u_{b_4} = 5$. The dual objective value is 14.
- We consider the set of edges E of the demand graph $E = \{(1,3),(2,1),(3,3),(4,3)\}$. Then $B^e = \{b_1, b_2, b_3, b_4\}$ and $G^e = G_0^e = \{g_1, g_3\}$. There does not exist a matching in this graph that allocates all the bidders in it.
- We formulate the dual of the restricted primal:

$$\begin{array}{lrcl} \max\limits_{u',p'} & -u'_{b_1} - u'_{b_2} - u'_{b_3} - u'_{b_4} - p'_{g_1} - p'_{g_3} & & \\ \text{s.t.} & u'_{b_1} + p'_{g_3} & \geq & 0 \\ & u'_{b_2} + p'_{g_1} & \geq & 0 \\ & u'_{b_3} + p'_{g_3} & \geq & 0 \\ & u'_{b_4} + p'_{g_3} & \geq & 0 \\ & u'_{b_1} + 1 & \geq & 0 \\ & u'_{b_2} + 1 & \geq & 0 \\ & u'_{b_3} + 1 & \geq & 0 \\ & u'_{b_4} + 1 & \geq & 0 \\ & p'_{g_1} & \geq & 0 \\ & p'_{g_3} & \geq & 0 \end{array}$$

- We know that $O = \{g_3\}$ is minimally over-demanded, and so we construct a feasible solution to the above program following the algorithm. Setting $p'_{g_3} = 1; u'_{b_1}, u'_{b_3}, u'_{b_4} = (-1)$ and every other variable to 0, the objective value of the program is 2. Then the negative of this is -2.

- We use this solution to increase the price of g_3 in the dual program by 1. Under these new prices we find a feasible dual solution. The utility of the bidders are now $u_i = \max_j(v_{ij} - p_j)$. We have $u_{b_1} = 2, u_{b_2} = 3, u_{b_3} = 2, u_{b_4} = 4$ and $p_{g_3} = 1$. The dual objective value is now 12.

This completes the first iteration of the primal-dual algorithm. We used the dual of the restricted primal to reduce the objective value of the dual. We now examine one more iteration of the algorithm.

- We consider the set of edges E of the demand graph $E = \{(1,2), (1,3), (2,1), (3,1), (3,3), (4,3)\}$. Then $B^e = \{b_1, b_2, b_3, b_4\}$ and $G^e = \{g_1, g_2, g_3\}$ but $G_0^e = \{g_1, g_2\}$. There does not exist a matching in this graph that allocates all the bidders in it.

- We formulate the dual of the restricted primal:

$$
\begin{array}{lrcl}
\max\limits_{u',p'} & -u'_{b_1} - u'_{b_2} - u'_{b_3} - u'_{b_4} - p'_{g_1} - p'_{g_2} - p'_{g_3} & & \\
\text{s.t.} & u'_{b_1} + p'_{g_2} & \geq & 0 \\
& u'_{b_1} + p'_{g_3} & \geq & 0 \\
& u'_{b_2} + p'_{g_1} & \geq & 0 \\
& u'_{b_3} + p'_{g_1} & \geq & 0 \\
& u'_{b_3} + p'_{g_3} & \geq & 0 \\
& u'_{b_4} + p'_{g_3} & \geq & 0 \\
& u'_{b_1} + 1 & \geq & 0 \\
& u'_{b_2} + 1 & \geq & 0 \\
& u'_{b_3} + 1 & \geq & 0 \\
& u'_{b_4} + 1 & \geq & 0 \\
& p'_{g_1} & \geq & 0 \\
& p'_{g_2} & \geq & 0 \\
& p'_{g_3} + 1 & \geq & 0
\end{array}
$$

- Note how the constraint for good g_3 changed because the price in the dual solution we started with was not non-zero. We know that $O = \{g_1, g_3\}$ is minimally over-demanded, and so we construct a feasible solution to the above program following the algorithm. Setting $p'_{g_1}, p'_{g_3} = 1; u'_{b_2}, u'_{b_3}, u'_{b_4} = (-1)$ and every other variable to 0, the objective value of the program is 1. Then the negative of this is -1.

- We use this solution to increase the prices of g_1 and g_3 in the dual program by 1. Under these new prices we find a feasible dual solution. The utility of the bidders are now $u_i = \max_j(v_{ij} - p_j)$. We have $u_{b_1} = 2, u_{b_2} = 2, u_{b_3} = 1, u_{b_4} = 3$ and prices $p_{g_1} = 1, p_{g_3} = 2$. The dual objective value is now 11.

This completes the second iteration of the primal-dual algorithm. We can now see that each iteration of the algorithm is exactly one price increase step of the auction. Then the primal-dual algorithm will also converge to the same matching and prices as the auction in one more iteration. The algorithm terminates with objective value 10 which is the maximum welfare as well as the sum of prices and utilities at the smallest Walrasian equilibrium.

Concluding Remarks. Today we saw a new perspective on ascending auctions based on linear programs and the primal-dual method. Several auctions in the literature have been formulated as primal-dual algorithms for diminishing marginal values, and gross substitutes, among others. For further reading refer to the discussion on primal-dual auctions by deVries et al [11].

[11] Sven de Vries, James Schummer, and Rakesh V Vohra. On ascending vickrey auctions for heterogeneous objects. *Journal of Economic Theory*, 132(1):95–118, 2007

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