Posted-Price Mechanisms

CS 1951k/2951z

2017-03-04

We present a method for choosing a posted price in a symmetric setting, along with a guarantee about the expected welfare of the method. Once again, this analysis demonstrates the tradeoff between optimal, complex mechanisms and approximately-optimal, simple mechanisms.

1 *Expected Welfare*

Imagine a set of n symmetric bidders,[1] with values drawn i.i.d. from some distribution F. The expected welfare of posted price π is the probability the good is allocated at price π (i.e., the probability some buyer's value exceeds π) times that buyer's expected value, given that its value exceeds π:

[1] buyers

$$W(\pi) = \Pr(\text{good is allocated at } \pi) \mathop{\mathbb{E}}_{v \sim F} [v \mid v \geq \pi] .$$

Since the probabilty that each buyer's value is less than π is $F(\pi)$, the probability the good is allocated at π is:

$$\begin{aligned} \Pr(\text{good is allocated at } \pi) &= \Pr(\text{some buyer's value is above } \pi) \\ &= 1 - \Pr(\text{no buyer's value is above } \pi) \\ &= 1 - \Pr(\text{all buyer's value are below } \pi) \\ &= 1 - \Pr(i\text{'s value is below } \pi)^n \\ &= 1 - F(\pi)^n. \end{aligned}$$

More formally,

$$\begin{aligned} \Pr(\text{good is allocated at } \pi) &= \Pr(\exists i \; s.t. \; v_i \geq \pi) \\ &= 1 - \Pr(\neg \exists i \; s.t. \; v_i \geq \pi) \\ &= 1 - \Pr(\forall i, \; v_i < \pi) \\ &= 1 - \Pr(v_i < \pi)^n \\ &= 1 - F(\pi)^n. \end{aligned}$$

Now, the welfare at posted price π is:

$$\begin{aligned} \mathop{\mathbb{E}}_{v \sim F} [v \mid v \geq \pi] &= \frac{\int_\pi^{\overline{v}} v f(v) \, \mathrm{d}v}{\Pr(v \geq \pi)} \\ &= \frac{\int_\pi^{\overline{v}} v f(v) \, \mathrm{d}v}{1 - F(\pi)}. \end{aligned}$$

Therefore, the expected welfare at posted price π is:

$$W(\pi) = (1 - F(\pi)^n) \left(\frac{\int_\pi^{\overline{v}} v f(v) \, \mathrm{d}v}{1 - F(\pi)} \right).$$

To maximize $W(\pi)$, we could compute the first derivative, and then find the posted price at which this derivative equals 0. As it turns out, this isn't any fun.[2] Consequently, we take an alternative approach in this lecture. We design a very simple mechanism that is a constant-approximation of the optimal posted-price mechanism.

[2] See Appendix B.

2 A Simple Posted-Price Mechanism

If the mechanism designer knew the buyers' values, then the optimal posted price would be the highest one. As it does not have access to this precise information, but only to distributions over their values, an intuitive way to set the posted price is such that, in expectation, exactly one buyer's value exceeds this price.

In a symmetric setting, this idea translates into setting π such that $\Pr(v \geq \pi) = 1/n$. Then, in expectation,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i \in N} \mathbb{1}_{v_i \geq \pi}\right] &= \sum_{i \in N} \mathbb{E}\left[\mathbb{1}_{v_i \geq \pi}\right] \\
&= \sum_{i \in N} \Pr(v_i \geq \pi) \\
&= n \Pr(v \geq \pi) \\
&= n\,(1/n) \\
&= 1
\end{aligned}$$

Indeed, in expectation, exactly one buyer's value exceeds π.

The posted price π^* such that $\Pr(v \geq \pi^*) = 1/n$ is $F^{-1}(1 - 1/n)$: i.e., $F(\pi^*) = 1 - 1/n$. In this lecture, we are interested in bounding the expected welfare of the mechanism that posts price π^*. We call this expected welfare APX. By symmetry,

$$\begin{aligned}
\text{APX} &= \Pr(\text{good is allocated at } \pi^*) \mathop{\mathbb{E}}_{v \sim F}\left[v \mid v \geq \pi^*\right] \\
&= \left(1 - F(\pi^*)^n\right) \mathop{\mathbb{E}}_{v \sim F}\left[v \mid v \geq \pi^*\right] \\
&= \left(1 - (1 - 1/n)^n\right) \mathop{\mathbb{E}}_{v \sim F}\left[v \mid v \geq \pi^*\right].
\end{aligned}$$

The optimal welfare in this setting can be achieved via a second-price auction. But we will derive a looser upper bound on the optimal posted-price mechanism, OPT, which relates to the lower bound we derive presently on APX.

Lemma 2.1. *The expected welfare of our simple posted-price mechanism, when all values are drawn i.i.d., is lower bounded as follows:*

$$APX \geq (1 - 1/e)\,\mathbb{E}\left[v \mid v \geq \pi^*\right].$$

Proof. Recall that the expected welfare generated by the simple posted-price mechanism is:

$$\mathrm{APX} = \left(1 - (1 - {}^1/_n)^n\right) \mathbb{E}\left[v \mid v \geq \pi^*\right].$$

To simplify this expression, we bound the term $(1 - {}^1/_n)^n$.

Let $f(n) = (1 - {}^1/_n)^n$. In the limit, as n tends toward infinity,[3]

$$\lim_{n \to \infty} f(n) = {}^1/_e.$$

Moveover, $f(n)$ is an increasing function when $n > 1$, as shown in Figure 1. It follows that

$$(1 - {}^1/_n)^n \leq {}^1/_e,$$

or equivalently,

$$1 - (1 - {}^1/_n)^n \geq 1 - {}^1/_e.$$

Therefore,

$$\begin{aligned} \mathrm{APX} &= \left(1 - (1 - {}^1/_n)^n\right) \mathbb{E}\left[v \mid v \geq \pi^*\right] \\ &\geq (1 - {}^1/_e)\, \mathbb{E}\left[v \mid v \geq \pi^*\right]. \end{aligned}$$

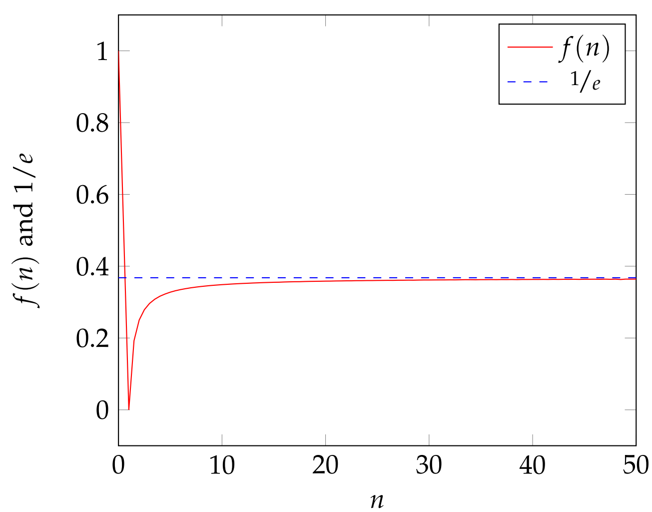

Figure 1: A comparison between $f(n) = (1 - {}^1/_n)^n$ and ${}^1/_e$. Observe that $f(n)$ is an increasing function when $n \geq 1$. Moreover, $f(n)$ approaches ${}^1/_e$ from below.

□

3 A Loose Upper Bound on OPT

We construct an upper bound on OPT by analyzing the total expected welfare of the second-price auction as compared to that of a posted-price mechanism that posts price $\pi^* = F^{-1}(1 - {}^1/_n)$ and sells n identical goods.[4] Assuming n goods, everyone can be served.

[3] If you are not satisfied with proof by picture, a formal proof of this inequality can be found in Appendix A.

[4] Equivalently, n simultaneous posted-price mechanisms, each of which posts price $\pi^* = F^{-1}(1 - {}^1/_n)$.

We argue that the welfare of this mechanism exceeds that of the second-price auction. As the welfare of the second-price auction is optimal, this reasoning yields an upper bound on the welfare of the optimal posted-price mechanism, OPT.

There are instances in which a single posted-price mechanism that posts price $\pi^* = F^{-1}(1 - 1/n)$ does not sell the good, because $v_{(1)} < \pi^*$, while an auction, which discovers a price, would. Nonetheless, we claim that the posted-price mechanism that sells n identical goods at posted price π^* always obtain more welfare, in expectation, than a second-price auction. The welfare obtained by selling additional copies of the good always makes up for the instances in which the single posted-price mechanism does not sell the good.

We do not prove this upper bound formally. Instead, we present a few (hopefully) insightful examples. In all our examples, there are $n = 2$ symmetric buyers, each of whom draws their value from a discrete distribution F, with two possible types. The low type occurs with probability q, and the high type, with probability $1 - q$.

Example 3.1. Suppose there are two types, $T = \{0, 1\}$, each drawn with equal probability: i.e., $q = 1 - q = 1/2$. There are four possible value profiles, all of which occur with equal probability:

$$\mathbf{v} = \begin{cases} (0,0), & \Pr(\mathbf{v}) = 1/4 \\ (0,1), & \Pr(\mathbf{v}) = 1/4 \\ (1,0), & \Pr(\mathbf{v}) = 1/4 \\ (1,1), & \Pr(\mathbf{v}) = 1/4. \end{cases}$$

- The expected welfare of a second-price auction is $3/4$.

$$1/4\,(0 + 1 + 1 + 1) = 3/4.$$

- The expected welfare of a posted-price mechanism with one good and posted price $\pi^* = F^{-1}(1/2) > 0$ is $3/4$.

$$1/4\,(0 + 1 + 1 + 1) = 3/4.$$

- The expected welfare of a posted-price mechanism with $n = 2$ identical copies of the good and posted price $\pi^* = F^{-1}(1/2) > 0$ is 1.

$$1/4\,(0 + 1 + 1 + 2) = 1.$$

This example shows that the total expected welfare of a posted-price mechanism with n goods can be strictly greater than the expected welfare of a second-price auction. We now show that these values can be made to be arbitrarily close.

Example 3.2. Suppose there are two types, $T = \{1-4\epsilon, 1\}$, for some $\epsilon \in (0,1]$, which occur with equal probability: i.e., $q = 1-q = 1/2$. There are four possible value profiles, all of which occur with equal probability:

$$\mathbf{v} = \begin{cases} (1-4\epsilon, 1-4\epsilon), & \Pr(\mathbf{v}) = 1/4 \\ (1-4\epsilon, 1), & \Pr(\mathbf{v}) = 1/4 \\ (1, 1-4\epsilon), & \Pr(\mathbf{v}) = 1/4 \\ (1,1), & \Pr(\mathbf{v}) = 1/4. \end{cases}$$

- The expected welfare of a second-price auction is

$$1/4\,(1-4\epsilon+1+1+1) = 1/4\,(4-4\epsilon) = 1-\epsilon.$$

- The expected welfare of a posted-price mechanism with one good and posted price $\pi^* = F^{-1}(1/2) > 1-4\epsilon$ is $3/4$.

$$1/4\,(0+1+1+1) = 3/4.$$

- The expected welfare of a posted-price mechanism with $n = 2$ identical copies of the good and posted price $\pi^* = F^{-1}(1/2) > 1-4\epsilon$ is $3/4$.

$$1/4\,(0+1+1+2) = 1.$$

Whereas the performance of the posted-price mechanism with one good matched that of the second-price auction in the first example, in this example, for small values of ϵ, the second-price auction generates strictly more expected welfare than the posted-price mechanism with one good. On the other hand, a posted-price mechanism with n copies of the good outperforms the second-price auction, for $\epsilon > 0$.

In both our examples so far, n posted-price mechanisms have outperformed the second-price auction (though not by much in the second example). How might the second-price auction outperform n posted-price mechanisms? The only possible way would be to shift mass to lower types which would not be allocated in the n posted-price mechanisms, but would be allocated in the second-price auction. However, when probability mass shifts to lower types, so, too, does the requisite posted price, $F^{-1}(1/2)$: i.e., the median. As a result, the second-price auction never strictly outperforms the posted-price mechanism that sells n identical copies of the good at this posted price. Our final example demonstrates this outcome, assuming a high probability of low-type profiles.

Example 3.3. Suppose there are two types, $T = \{1-\epsilon, 1\}$, for some $\epsilon \in (0,1]$, and with probability $q \gg 1-q$, a buyer has type $1-\epsilon$.

As usual, there are four possible value profiles, but now the profile in which both buyers have low types is significantly more probable:

$$\mathbf{v} = \begin{cases} (1-\epsilon, 1-\epsilon), & \Pr(\mathbf{v}) = q^2 \\ (1-\epsilon, 1), & \Pr(\mathbf{v}) = q(1-q) \\ (1, 1-\epsilon), & \Pr(\mathbf{v}) = (1-q)q \\ (1,1), & \Pr(\mathbf{v}) = (1-q)^2. \end{cases}$$

- The expected welfare of a second-price auction is

$$\left[q^2\right](1-\epsilon) + 2\left[q(1-q)\right](1) + \left[(1-q)^2\right](1).$$

- The expected welfare of a posted-price mechanism with one good and posted price be $\pi^* = F^{-1}(1/2) \leq 1-\epsilon$ is

$$\left[q^2\right](1-\epsilon) + 2\left[q(1-q)\right]\left(\frac{(1-\epsilon)+1}{2}\right) + \left[(1-q)^2\right](1).$$

- The expected welfare of a posted-price mechanism with $n = 2$ identical copies of the good and posted price be $\pi^* = F^{-1}(1/2) \leq 1-\epsilon$ is

$$\left[q^2\right](2-2\epsilon) + 2\left[q(1-q)\right](2-\epsilon) + \left[(1-q)^2\right](2).$$

Comparing the terms for the posted-price mechanism that sells n identical copies of the good with the terms for the second-price auction, we see that each term in the former is at least that of the corresponding term in the latter.

Based on this series of examples, we make the following claim (without proof):

Proposition 3.4. *The welfare of the second-price auction is upper-bounded by the welfare of the posted-price mechanism that sells n identical copies of the good at posted price* $\pi^* = F^{-1}(1 - 1/n)$.

Corollary 3.5. *The welfare of the optimal posted-price mechanism is upper-bounded by the welfare of a posted-price mechanism that sells n identical copies of the good at posted price* $\pi^* = F^{-1}(1 - 1/n)$.

Proof. The welfare of the optimal posted-price mechanism is upper-bounded by the welfare of the second-price auction. □

Using this corollary, we derive an upper bound on OPT, the welfare of the optimal posted-price mechanism.

Lemma 3.6. *The welfare of the optimal posted-price mechanism is upper-bounded as follows:*

$$OPT \leq \mathbb{E}\left[v \mid v \geq \pi^*\right],$$

where $\pi^* = F^{-1}(1 - 1/n)$.

Proof. By Corollary 3.5, it suffices to calculate the expected welfare of a posted-price mechanism that can sell up to n copies of the good. Since buyers do not compete with one another in this mechanism, we can consider each buyer's impact on welfare independently of any others'. Hence, the expected welfare of this mechanism is:

$$\begin{aligned}
\mathrm{OPT} &\leq \sum_{i=1}^{n} \Pr\left(v_i \geq \pi^*\right) \mathbb{E}\left[v_i \mid v_i \geq \pi^*\right] \\
&= n \Pr\left(v \geq \pi^*\right) \mathbb{E}\left[v \mid v \geq \pi^*\right] \\
&= n \left(1/n\right) \mathbb{E}\left[v \mid v \geq \pi^*\right] \\
&= \mathbb{E}\left[v \mid v \geq \pi^*\right]
\end{aligned}$$

The second equality follows from the symmetry assumption, and the third, from the fact that $\pi^* = F^{-1}\left(1 - 1/n\right)$, so $F(\pi^*) = 1 - 1/n$, which implies $\Pr\left(v \geq \pi^*\right) = 1/n$. □

4 An Approximation Ratio

We can now derive an approximation ratio for the total expected welfare of our simple posted-price mechanism, assuming the buyers' values are drawn i.i.d. from distribution F.

Theorem 4.1. *The approximation ratio of the welfare generated by the posted-price mechanism that posts price $\pi^* = F^{-1}\left(1 - 1/n\right)$ is*

$$\frac{APX}{OPT} \geq 1 - 1/e,$$

assuming all values are i.i.d..

Proof. Start with the lower bound:

$$\mathrm{APX} \geq \left(1 - 1/e\right) \mathbb{E}\left[v \mid v \geq \pi^*\right].$$

Divide by OPT:

$$\frac{\mathrm{APX}}{\mathrm{OPT}} \geq \left(1 - 1/e\right) \frac{\mathbb{E}\left[v \mid v \geq \pi^*\right]}{\mathrm{OPT}}.$$

But $\mathrm{OPT} \leq \mathbb{E}\left[v \mid v \geq \pi^*\right]$, so

$$\begin{aligned}
\frac{\mathrm{APX}}{\mathrm{OPT}} &\geq \left(1 - 1/e\right) \frac{\mathbb{E}\left[v \mid v \geq \pi^*\right]}{\mathbb{E}\left[v \mid v \geq \pi^*\right]} \\
&= 1 - 1/e \\
&\geq 0.63.
\end{aligned}$$

□

This result is surprising. What it means is that by posting a price of $F^{-1}(1 - 1/n)$, we can obtain at least 0.63 of the optimal expected welfare. One way to interpret this result is that competition, which is downplayed by this mechanism but is a key ingredient of auctions, accounts for at most 0.37 of the optimal expected welfare.

A The Exponential Function

Let
$$f(x, n) = (1 + x/n)^n .$$
We investigate this function in the limit, as n approaches ∞, by taking logs:
$$\begin{aligned}\log\left(\lim_{n\to\infty}(1 + x/n)^n\right) &= \lim_{n\to\infty} n \log(1 + x/n) \\ &= \lim_{n\to\infty} \frac{\log(1 + x/n)}{1/n}.\end{aligned}$$
BY L'Hôpital's rule,[5]
$$\begin{aligned}\lim_{n\to\infty} \frac{\log(1 + x/n)}{1/n} &= \lim_{n\to\infty} \frac{[\log(1 + x/n)]'}{[1/n]'} \\ &= \lim_{n\to\infty} \frac{-x/n^2}{(1 + x/n)(-1/n^2)} \\ &= \lim_{n\to\infty} \frac{x}{1 + x/n} \\ &= x.\end{aligned}$$
Since
$$\log\left(\lim_{n\to\infty}(1 + x/n)^n\right) = x,$$
it follows that
$$\lim_{n\to\infty}(1 + x/n)^n = e^x.$$
Letting $x = -1$ yields
$$\lim_{n\to\infty}(1 - 1/n)^n = e^{-1}.$$
By taking logs, we restricted the domain of f to $n > 1$, because when $n = 1$, $\log(1 - 1/n) = -\infty$. For $n > 1$, $\log(1 - 1/n)$ is an increasing function, and so is $n \log(1 - 1/n) = \log f(-1, n)$. Finally, since log itself is increasing, $f(-1, n)$ is also increasing. Therefore,
$$(1 - 1/n)^n \leq 1/e.$$

[5] $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

B The First Derivative of the Expected Welfare

Observe the following:

$$\begin{aligned}
\left[1 - F(\pi)^n\right]' &= -nF(\pi)^{n-1} f(\pi) \\
\left[(1 - F(\pi))^{-1}\right]' &= -(1 - F(\pi))^{-2}(-f(\pi)) \\
&= (1 - F(\pi))^{-2} f(\pi) \\
\left[\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v\right]' &= -\pi f(\pi).
\end{aligned}$$

Hence, the first derivative of the expected welfare $W(\pi)$ is:

$$\begin{aligned}
\frac{\mathrm{d}W(\pi)}{\mathrm{d}\pi} &= \frac{\mathrm{d}\left(\left(1 - F(\pi)^n\right) \frac{\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v}{1 - F(\pi)}\right)}{\mathrm{d}\pi} \\
&= \left[1 - F(\pi)^n\right]' \left[(1 - F(\pi))^{-1}\right] \left[\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v\right] \\
&\quad + \left[1 - F(\pi)^n\right] \left[(1 - F(\pi))^{-1}\right]' \left[\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v\right] \\
&\quad + \left[1 - F(\pi)^n\right] \left[(1 - F(\pi))^{-1}\right] \left[\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v\right]' \\
&= -\left[nF(\pi)^{n-1} f(\pi)\right] \left[(1 - F(\pi))^{-1}\right] \left[\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v\right] \\
&\quad + \left[1 - F(\pi)^n\right] \left[(1 - F(\pi))^{-2} f(\pi)\right] \left[\int_{\pi}^{\overline{v}} v f(v)\,\mathrm{d}v\right] \\
&\quad - \left[1 - F(\pi)^n\right] \left[(1 - F(\pi))^{-1}\right] \left[\pi f(\pi)\right]
\end{aligned}$$

We stopped having fun at this point. But please feel free to set this derivative equal to 0, and solve for the optimal posted price. Is the resulting mechanism interpretable? If so, definitely let us know!