Welfare-Maximizing Auctions

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We present a mathematical program and a pointwise approach to maximizing welfare in single-parameter auctions.

1 *Mathematical Program*

Recall that we can pose a single-parameter auction design problem as a mathematical program with the following constraints:

1. incentive compatibility
2. individual rationality
3. allocation constraints
4. ex-post feasibility

More specifically, if our objective is to maximize welfare, this program can be stated formally as follows:

$$\begin{aligned}
&\max_{\mathbf{x},\mathbf{p}} \mathop{\mathbb{E}}_{\mathbf{v}\sim F}\left[\sum_{i\in N} v_i x(v_i,\mathbf{v}_{-i})\right] \\
\text{subject to } & v_i x_i(v_i,\mathbf{v}_{-i}) - p_i(v_i,\mathbf{v}_{-i}) \geq v_i x_i(t_i,\mathbf{v}_{-i}) - p_i(t_i,\mathbf{v}_{-i}) && \forall i \in N, \forall \mathbf{v} \in T \\
& v_i x_i(v_i,\mathbf{v}_{-i}) - p_i(v_i,\mathbf{v}_{-i}) \geq 0 && \forall i \in N, \forall \mathbf{v} \in T \\
& 0 \leq x_i(v_i,\mathbf{v}_{-i}) \leq 1 && \forall i \in N, \forall \mathbf{v} \in T \\
& \sum_{i\in N} x_i(v_i,\mathbf{v}_{-i}) \leq 1 && \forall \mathbf{v} \in T
\end{aligned}$$

2 *Myerson's Auction Design Recipe*

Myerson's lemma tells us that for single-parameter auctions, we can re-express the IC and IR constraints in terms of monotonicity of the allocation rule and a particular payment rule instead:

$$\begin{aligned}
& x_i(v_i,\mathbf{v}_{-i}) \geq x_i(t_i,\mathbf{v}_{-i}), && \forall i \in N, \forall v_i \geq t_i \in T_i \\
& p_i(v_i,\mathbf{v}_{-i}) = v_i x_i(v_i,\mathbf{v}_{-i}) - \int_0^{v_i} x_i(z,\mathbf{v}_{-i})\,\mathrm{d}z,, && \forall i \in N, \forall \mathbf{v} \in T.
\end{aligned}$$

Myerson's lemma therefore suggests a two-step process (i.e., a design recipe) to solving for a welfare-maximizing auction:

1. First, design an algorithm that can find an ex-post feasible, welfare-maximizing allocation given $\mathbf{v} \in T$.

2. If it so happens that monotonicity holds, construct truthful payments using Myerson's payment formula.

A welfare-maximization algorithm that relies on Myerson's Lemma is given in Algorithm 1. Allocations are solved for first (using the greedy algorithm, which is monotone), based on which appropriate payments are constructed.

Observe that this auction is **prior-free**. No knowledge of the value distributions is needed to implement a welfare-maximizing auction. Expected total welfare is maximized by maximizing welfare pointwise, the latter being a more stringest objective.

Algorithm 1: Welfare Maximization. This algorithm proceeds **pointwise**, meaning one value vector at a time.

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for all $\mathbf{v} \in T$ **do** ▷ Find allocations
    for all $i \in N$ **do**
        $x_i(v_i, \mathbf{v}_{-i}) \leftarrow 0$
    end for
    if $\max_i\{v_i : i \in N\} > 0$ **then**
        $w(\mathbf{v}) \leftarrow \arg\max_i\{v_i : i \in N\}$
        for all $i^* \in w(\mathbf{v})$ **do**
            $x_{i^*}(v_{i^*}, \mathbf{v}_{-i^*}) \leftarrow 1/|w(\mathbf{v})|$
        end for
    end if
end for
for all $i \in N$ **do** ▷ Compute payments
    for all $\mathbf{v} \in T$ **do**
        $p_i(v_i, \mathbf{v}_{-i}) \leftarrow v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z$
    end for
end for
$S \leftarrow \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim F}\left[v_i x_i(v_i, \mathbf{v}_{-i})\right]$ ▷ Total expected welfare
$R \leftarrow \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim F}\left[p_i(v_i, \mathbf{v}_{-i})\right]$ ▷ Total expected revenue
return S, R, $\mathbf{x}$, $\mathbf{p}$
