Revenue-Maximizing Auctions

CS 1951k/2951z

2020-02-26

We present a mathematical program and a pointwise approach to solving for optimal (i.e., revenue-maximizing) single-parameter auctions.

1 *Mathematical Program*

Recall that we can pose a single-parameter auction design problem as a mathematical program with the following constraints:

1. incentive compatibility
2. individual rationality
3. allocation constraints
4. ex-post feasibility

More specifically, if our objective is to maximize revenue, this program can be stated formally as follows:

$$\max_{\mathbf{x},\mathbf{p}} \mathop{\mathbb{E}}_{\mathbf{v}\sim F}\left[\sum_{i\in N} p(v_i,\mathbf{v}_{-i})\right]$$

$$\begin{aligned}
\text{subject to } & v_i x_i(v_i,\mathbf{v}_{-i}) - p_i(v_i,\mathbf{v}_{-i}) \geq v_i x_i(t_i,\mathbf{v}_{-i}) - p_i(t_i,\mathbf{v}_{-i}) && \forall i \in N, \forall \mathbf{v} \in T \\
& v_i x_i(v_i,\mathbf{v}_{-i}) - p_i(v_i,\mathbf{v}_{-i}) \geq 0 && \forall i \in N, \forall \mathbf{v} \in T \\
& 0 \leq x_i(v_i,\mathbf{v}_{-i}) \leq 1 && \forall i \in N, \forall \mathbf{v} \in T \\
& \sum_{i\in N} x_i(v_i,\mathbf{v}_{-i}) \leq 1 && \forall \mathbf{v} \in T
\end{aligned}$$

2 *Myerson's Auction Design Recipe*

Myerson's lemma tells us that we can re-express the IC and IR constraints in terms of monotonicity of the allocation rule and a particular payment rule instead:

$$\begin{aligned}
& x_i(v_i,\mathbf{v}_{-i}) \geq x_i(t_i,\mathbf{v}_{-i}), && \forall i \in N, \forall v_i \geq t_i \in T_i \\
& p_i(v_i,\mathbf{v}_{-i}) = v_i x_i(v_i,\mathbf{v}_{-i}) - \int_0^{v_i} x_i(z,\mathbf{v}_{-i})\,\mathrm{d}z,, && \forall i \in N, \forall \mathbf{v} \in T.
\end{aligned}$$

Myerson's theorem tells us that we can replace the objective of maximizing total expected payments with the equivalent objective of maximizing total expected virtual welfare.

Analogous to the case of welfare, Myerson's results suggest a two-step process (i.e., a design recipe) to solving for a revenue-maximizing auction:

1. First, design an algorithm that can find an ex-post feasible, virtual-welfare-maximizing allocation given $\mathbf{v} \in T$.

2. If it so happens that monotonicity holds, construct truthful payments using Myerson's payment formula.

An allocation scheme that maximizes virtual welfare allocates in non-increasing order of virtual values. The monotonicity constraint, however, applies to values, not virtual values: i.e., as a bidder's value increases, their probability of being allocated should not decrease. Thus, monotonicity holds precisely when the virtual value functions are all non-decreasing in values. When this property of a virtual value function holds, the underlying distribution in terms of which it is defined is said to be **regular**.

Algorithm 1: Revenue Maximization. This algorithm proceeds **pointwise**, meaning one value vector at a time.

---

for all $\mathbf{v} \in T$ **do** ▷ Find allocations
    for all $i \in N$ **do**
        $x_i(v_i, \mathbf{v}_{-i}) \leftarrow 0$
    end for
    if $\max_i\{\varphi_i(v_i) : i \in N\} \geq 0$ **then**
        $w(\mathbf{v}) \leftarrow \arg\max_i\{\varphi_i(v_i) : i \in N\}$
        for all $i^* \in w(\mathbf{v})$ **do**
            $x_{i^*}(v_{i^*}, \mathbf{v}_{-i^*}) \leftarrow 1/|w(\mathbf{v})|$
        end for
    end if
end for
for all $i \in N$ **do** ▷ Compute payments
    for all $\mathbf{v} \in T$ **do**
        $p_i(v_i, \mathbf{v}_{-i}) \leftarrow v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z$
    end for
end for
$S \leftarrow \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim F}\left[v_i x_i(v_i, \mathbf{v}_{-i})\right]$ ▷ Total expected welfare
$R \leftarrow \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim F}\left[p_i(v_i, \mathbf{v}_{-i})\right]$ ▷ Total expected revenue
return S, R, $\mathbf{x}$, $\mathbf{p}$
