Myerson's Payment Characterization

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We describe Myerson's lemma, in which he characterizes the payment rule that incentivizes truth telling in single-parameter auctions.

1 *Payment Characterization*

Theorem 1.1 (Myerson 1981)**.** *A single-parameter auction that allocates nothing to a lowest type (which, for simplicity, we assume is zero, for all bidders) satisfies incentive compatibility (IC) and individual rationality (IR) if and only if the following two conditions hold:*

1. *The allocation rule is monotone:*

$$x_i(v_i, \mathbf{v}_{-i}) \geq x_i(t_i, \mathbf{v}_{-i}), \quad \forall i \in N, \forall v_i \geq t_i \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \tag{1}$$

2. *Payments are computed as follows:*

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z, \forall i \in N, \forall v_i \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \tag{2}$$

Myerson's payment formula, Equation (2), is easy to interpret by visualizing it. The payment at a point v_i is simply the area to the left of the allocation function at $x_i(v_i, \mathbf{v}_{-i})$.

Begin by drawing a box $v_i x_i(v_i, \mathbf{v}_{-i})$, as in Figure 1. Next, remove the area under the allocation curve, namely $\int_0^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z$, which is depicted in Figure 2. The remaining area is the payment bidder i makes, as in Figure 3.

Proof. We first prove the if direction: i.e., incentive compatibility and individual rationality imply that the allocation rule is monotone, and payments take the form of Equation (2).

First, we show that incentive compatibility implies that the allocation rule must be monotone non-decreasing. By incentive compatibility, $\forall i \in N$, $\forall \mathbf{v}_{-i} \in T_{-i}$, and for any two types $v_i, t_i \in T_i$:

$$\begin{aligned} u_i(v_i, \mathbf{v}_{-i}) &\geq u_i(t_i, \mathbf{v}_{-i}) \\ u_i(t_i, \mathbf{v}_{-i}) &\geq u_i(v_i, \mathbf{v}_{-i}). \end{aligned}$$

Equivalently,

$$\begin{aligned} v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}) &\geq v_i x_i(t_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \\ t_i x_i(t_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) &\geq t_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}). \end{aligned}$$

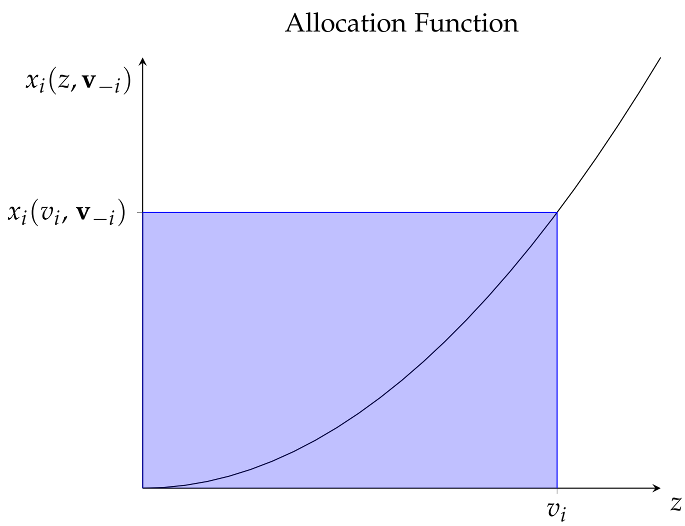

Figure 1: Area $v_i x_i(v_i, \mathbf{v}_{-i})$.

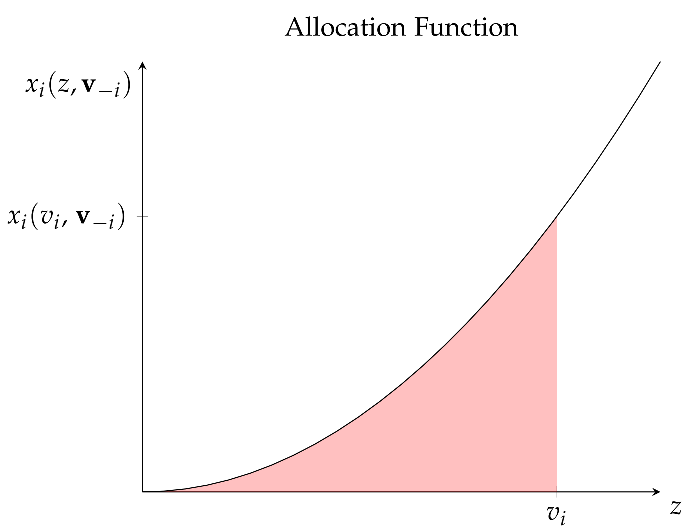

Figure 2: Area under the allocation curve, $\int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, dz$.

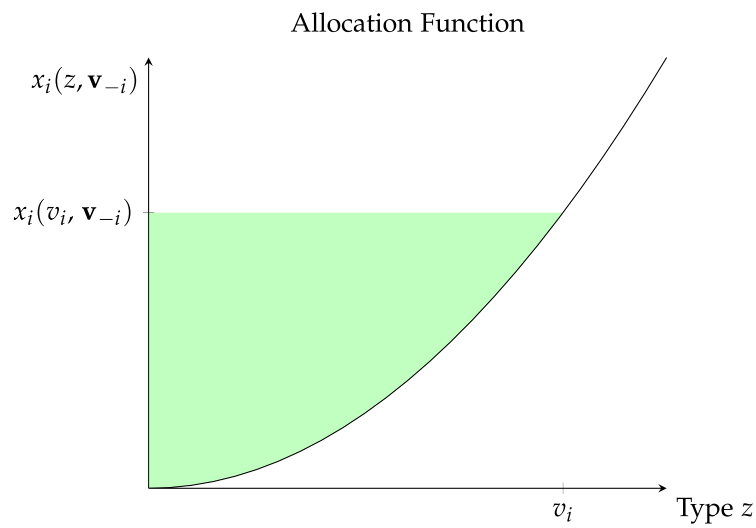

Figure 3: Payment, $v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z$.

Rearrange the expressions to collect payments:

$$\begin{aligned} v_i x_i(v_i, \mathbf{v}_{-i}) - v_i x_i(t_i, \mathbf{v}_{-i}) &\geq p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \\ p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) &\geq t_i x_i(v_i, \mathbf{v}_{-i}) - t_i x_i(t_i, \mathbf{v}_{-i}). \end{aligned}$$

Combine the expressions to form one inequality:

$$v_i x_i(v_i, \mathbf{v}_{-i}) - v_i x_i(t_i, \mathbf{v}_{-i}) \geq p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \geq t_i x_i(v_i, \mathbf{v}_{-i}) - t_i x_i(t_i, \mathbf{v}_{-i}).$$

Simplify the expression by collecting like terms:

$$v_i \left(x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i}) \right) \geq p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \geq t_i \left(x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i}) \right). \tag{3}$$

If $v_i \geq t_i$, then in order for this inequality to hold, $x_i(v_i, \mathbf{v}_{-i})$ cannot be less than $x_i(t_i, \mathbf{v}_{-i})$. So, the allocation rule must be monotone.

Next, we show that payments must take the form of Equation (2). Continuing where we left off, we divide each expression by $v_i - t_i$:

$$v_i \left(\frac{x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{v_i - t_i} \right) \geq \left(\frac{p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i})}{v_i - t_i} \right) \geq t_i \left(\frac{x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{v_i - t_i} \right).$$

If $v_i \geq t_i$, then we can write v_i as $v_i = t_i + \delta$, for some $\delta \geq 0$:

$$(t_i + \delta) \left(\frac{x_i(t_i + \delta, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{t_i + \delta - t_i} \right) \geq \frac{p_i(t_i + \delta, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i})}{t_i + \delta - t_i} \geq t_i \left(\frac{x_i(t_i + \delta, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{t_i + \delta - t_i} \right).$$

Now that we have functions of form

$$\frac{f(x + \delta) - f(x)}{\delta},$$

we can take derivatives by observing what happens in the limit:

$$f'(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}.$$

In the limit, as δ approaches 0, we can relate the derivative of the allocation function to the derivative of the payment function:

$$z\left(\frac{\mathrm{d}x_i(z,\mathbf{v}_{-i})}{\mathrm{d}z}\right)=\frac{\mathrm{d}p_i(z,\mathbf{v}_{-i})}{\mathrm{d}z}.$$

We integrate both sides from the lowest type, $z=0$, to $z=v_i$ to get

$$\begin{aligned}\int_0^{v_i} z\left(\frac{\mathrm{d}x_i(z,\mathbf{v}_{-i})}{\mathrm{d}z}\right)\mathrm{d}z &= \int_0^{v_i}\frac{\mathrm{d}p_i(z,\mathbf{v}_{-i})}{\mathrm{d}z}\,\mathrm{d}z\\ &= p_i(v_i,\mathbf{v}_{-i})-p_i(0,\mathbf{v}_{-i})\\ &= p_i(v_i,\mathbf{v}_{-i}).\end{aligned}$$

(By assumption, a bidder of type $v_i=0$ is never allocated; hence, by individual rationality, such a bidder must pay zero.)

This final equality is actually equivalent to Equation (2). In order to make the forms match, we integrate the left-hand side by parts:

$$\int_a^b u\,\mathrm{d}v = uv|_a^b-\int_a^b v\,\mathrm{d}u,$$

where we let

$$\begin{aligned} u &= z & \mathrm{d}u &= \mathrm{d}z\\ \mathrm{d}v &= \frac{\mathrm{d}x_i(z,\mathbf{v}_{-i})}{\mathrm{d}z}\mathrm{d}z & v &= x_i(z,\mathbf{v}_{-i}),\end{aligned}$$

to get:

$$\begin{aligned}\int_0^{v_i} z\left(\frac{\mathrm{d}x_i(z,\mathbf{v}_{-i})}{\mathrm{d}z}\right)\mathrm{d}z &= zx_i(z,\mathbf{v}_{-i})|_0^{v_i}-\int_0^{v_i}x_i(z,\mathbf{v}_{-i})\,\mathrm{d}z\\ &= v_ix_i(v_i,\mathbf{v}_{-i})-\int_0^{v_i}x_i(z,\mathbf{v}_{-i})\,\mathrm{d}z.\end{aligned}$$

We now prove the only if direction: i.e., that if the allocation rule is monotone and payments take the form of Equation (2), then incentive compatibility and individual rationality must hold.

First, we show that individual rationality holds. The utility of each bidder $i\in N$ by using the payment rule, Equation (2), with a monotone allocation rule, is

$$\begin{aligned}u_i(v_i,\mathbf{v}_{-i}) &= v_ix_i(v_i,\mathbf{v}_{-i})-p_i(v_i,\mathbf{v}_{-i})\\ &= v_ix_i(v_i,\mathbf{v}_{-i})-\left(v_ix_i(v_i,\mathbf{v}_{-i})-\int_0^{v_i}x_i(z,\mathbf{v}_{-i})\,\mathrm{d}z\right)\\ &= \int_0^{v_i}x_i(z,\mathbf{v}_{-i})\,\mathrm{d}z\\ &\geq 0.\end{aligned}$$

Now we show that incentive compatibility also holds: Bidding neither $v_i+\delta$ nor $v_i-\delta$, for some $\delta>0$, is preferable to bidding v_i.

By bidding $v_i + \delta$ for some $\delta > 0$, bidder i's utility is:

$$\begin{aligned}
u_i(v_i+\delta, \mathbf{v}_{-i}; v_i) &= v_i x_i(v_i+\delta, \mathbf{v}_{-i}) - p_i(v_i+\delta, \mathbf{v}_{-i}) \\
&= v_i x_i(v_i+\delta, \mathbf{v}_{-i}) - \left((v_i+\delta)\, x_i(v_i+\delta, \mathbf{v}_{-i}) - \int_0^{v_i+\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \right) \\
&= -\delta x_i(v_i+\delta, \mathbf{v}_{-i}) + \int_0^{v_i+\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z.
\end{aligned}$$

Comparing utilities between a truthful bid and any higher bid, we have:

$$\begin{aligned}
&\int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z - \left[-\delta x_i(v_i+\delta, \mathbf{v}_{-i}) + \int_0^{v_i+\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \right] \\
&= \delta x_i(v_i+\delta, \mathbf{v}_{-i}) - \int_{v_i}^{v_i+\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \\
&\geq 0.
\end{aligned}$$

The inequality follows from the monotonicty of the allocation function. For all $\gamma \in [v_i, v_i + \delta]$, $x_i(\gamma, \mathbf{v}_{-i}) \leq x_i(v_i + \delta, \mathbf{v}_{-i})$. Therefore, the integral is upper-bounded by $\delta x_i(v_i + \delta, \mathbf{v}_{-i})$. See Figure 4.

The situation is analogous for $v_i - \delta$. By bidding this amount, bidder i's utility is:

$$\begin{aligned}
u_i(v_i-\delta, \mathbf{v}_{-i}; v_i) &= v_i x_i(v_i-\delta, \mathbf{v}_{-i}) - p_i(v_i-\delta, \mathbf{v}_{-i}) \\
&= v_i x_i(v_i, \mathbf{v}_{-i}) - \left((v_i-\delta)\, x_i(v_i+\delta, \mathbf{v}_{-i}) - \int_0^{v_i-\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \right) \\
&= \delta x_i(v_i-\delta, \mathbf{v}_{-i}) + \int_0^{v_i-\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z.
\end{aligned}$$

Comparing utilities between a truthful bid and any lower bid, we have:

$$\begin{aligned}
&\int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z - \left[\delta x_i(v_i-\delta, \mathbf{v}_{-i}) + \int_0^{v_i-\delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \right] \\
&= \int_{v_i-\delta}^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z - \delta x_i(v_i-\delta, \mathbf{v}_{-i}) \\
&\geq 0.
\end{aligned}$$

The inequality follows from the monotonicty of the allocation function. For all $\gamma \in [v_i - \delta, v_i]$, $x_i(\gamma, \mathbf{v}_{-i}) \geq x_i(v_i - \delta, \mathbf{v}_{-i})$. Therefore, the integral is lower-bounded by $\delta x_i(v_i - \delta, \mathbf{v}_{-i})$. See Figure 4.

Since $\delta = 0$ is optimal, we have incentive compatibility.

$\square$

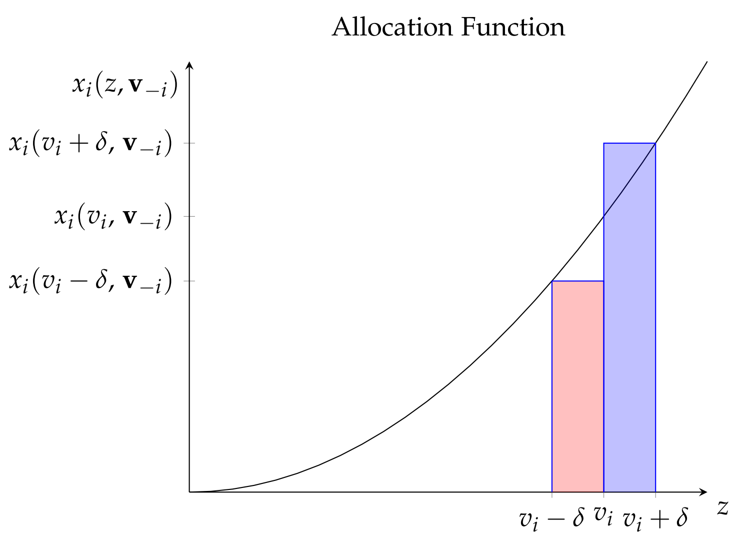

Figure 4: Bidding truthfully vs. not. Bidding truthfully is undominated.