Linear Programming

CSCI 1951k/2951z

Linear Programming

1. Preliminaries
2. Algorithms
3. Duality
4. Extensions

Preliminaries

What is linear programming?

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   a. *f* is a linear function of *x*
   b. *A* consists of linear inequality constraints of *x*

2. Written in the following form:

$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ \text{and} & \mathbf{x} \geq \mathbf{0} \end{array}$$

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4. b is an *m* by 1 vector of constraint bounds

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1. There are 2 possible inputs, gold and diamonds.
2. There are 2 possible outputs, rings and necklaces.

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1. Rings require 3 units of gold, 1 unit of diamond.
2. Necklaces require 2 units of gold, 3 units of diamond.
3. Rings sell for 300$, necklaces sell for 500$.
4. You have 22 units of gold, 12 units of diamond.

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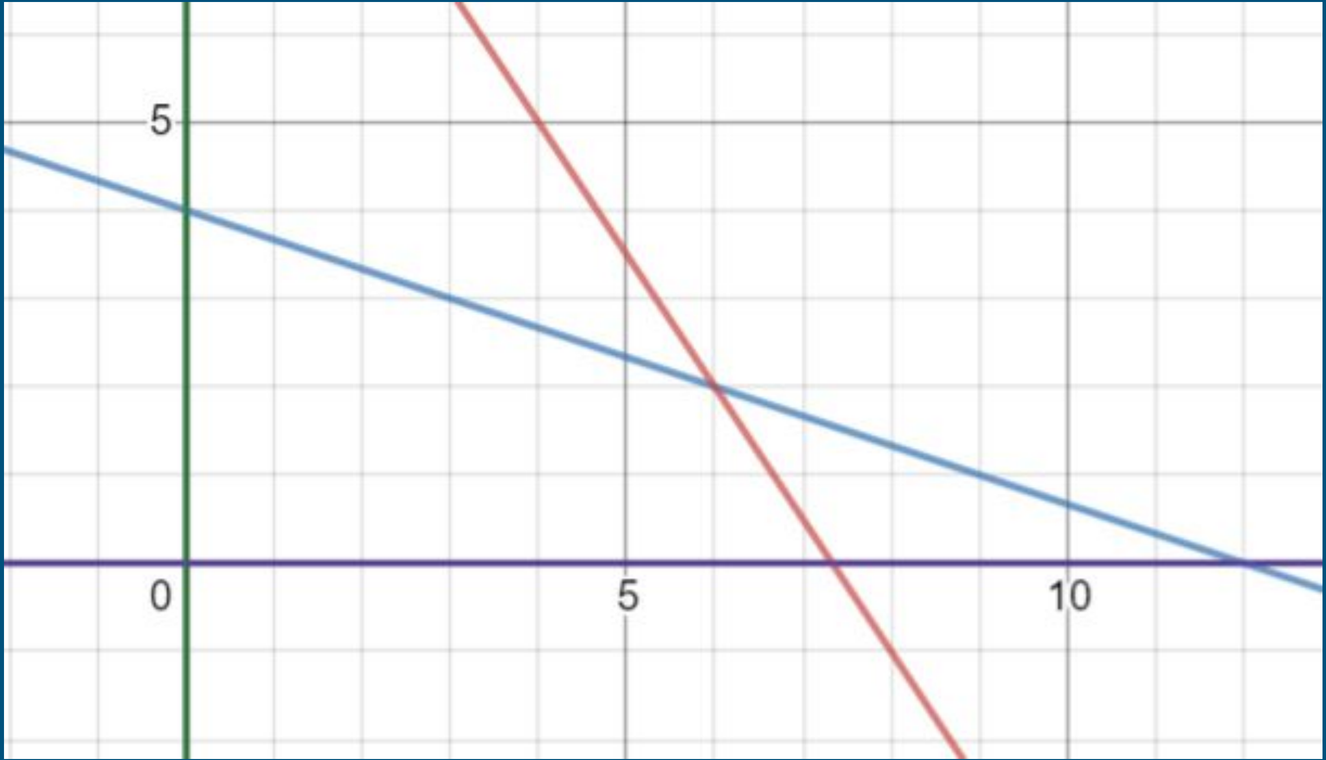
Can you find the optimal values of x?

$x = (6, 2) \rightarrow$ Revenue = 6 * 300 + 2 * 500 = 2800$

Preliminaries

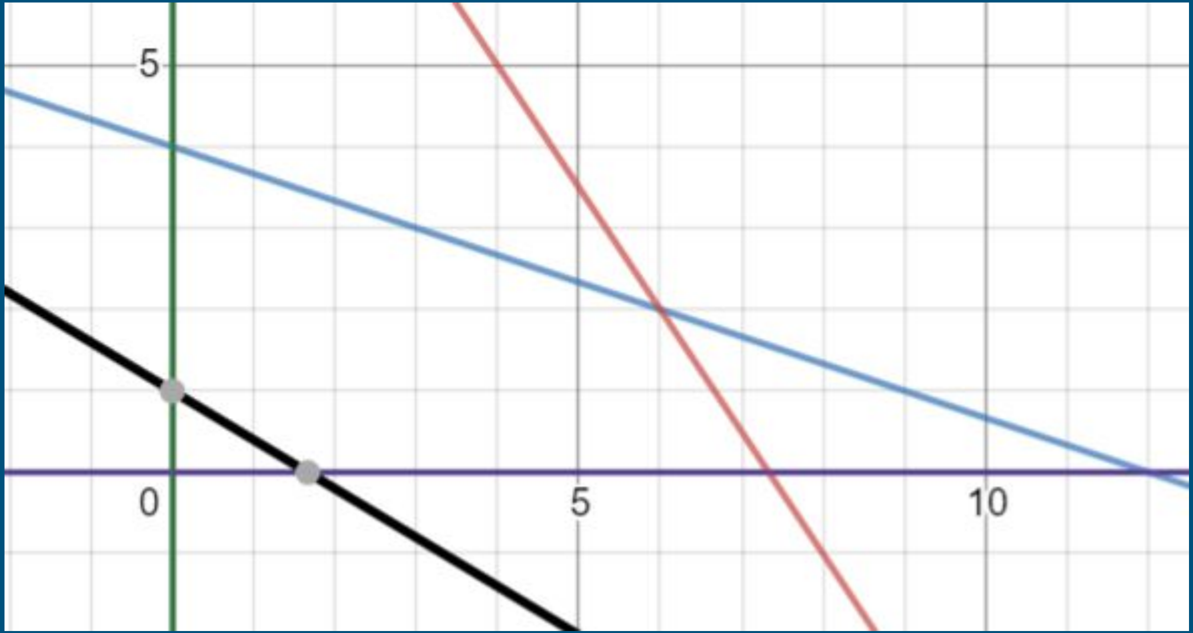


Here is the feasible region

Interesting to note that the optimal solution (6, 2) is at the intersection of two constraints.

Preliminaries

Why does this happen? Let's plot the objective function.



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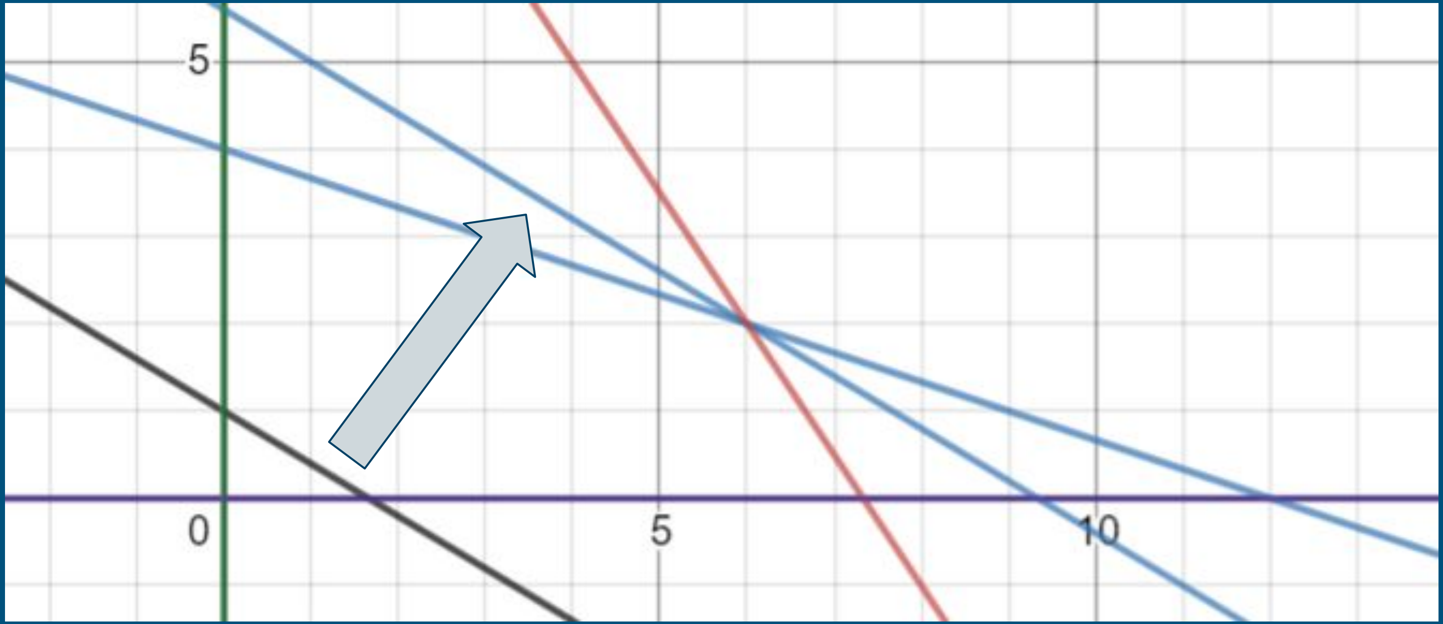

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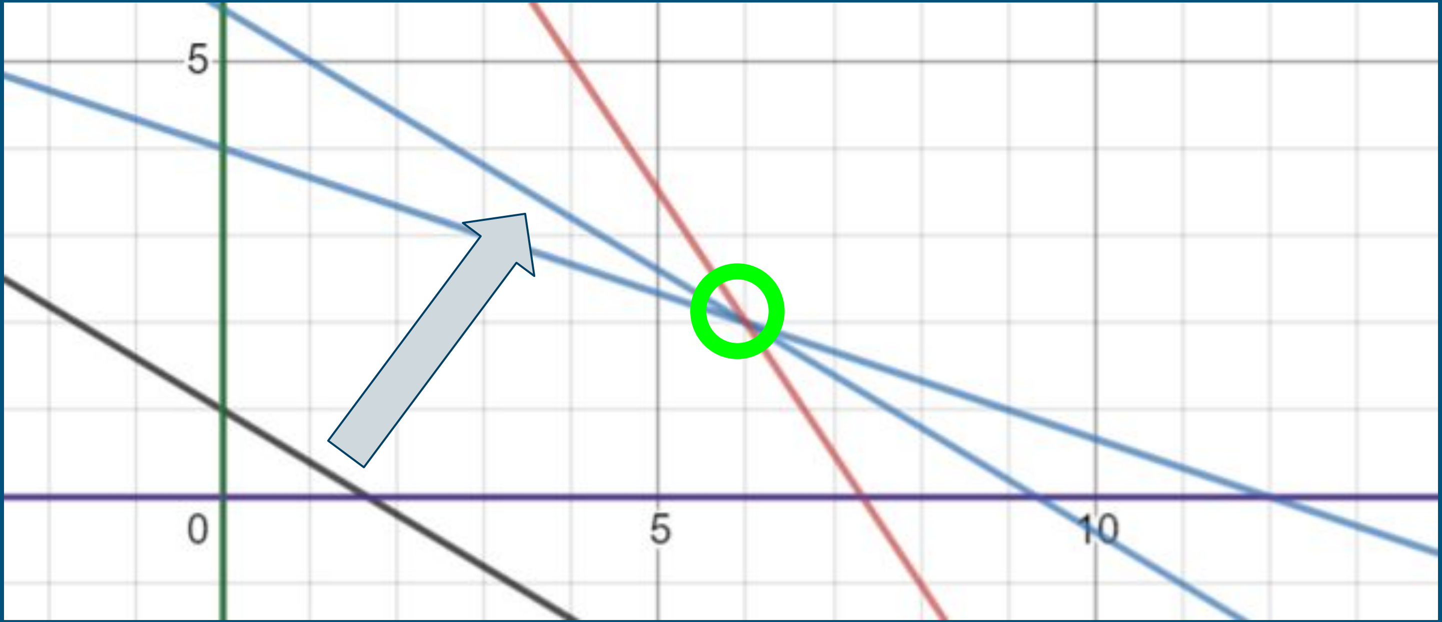
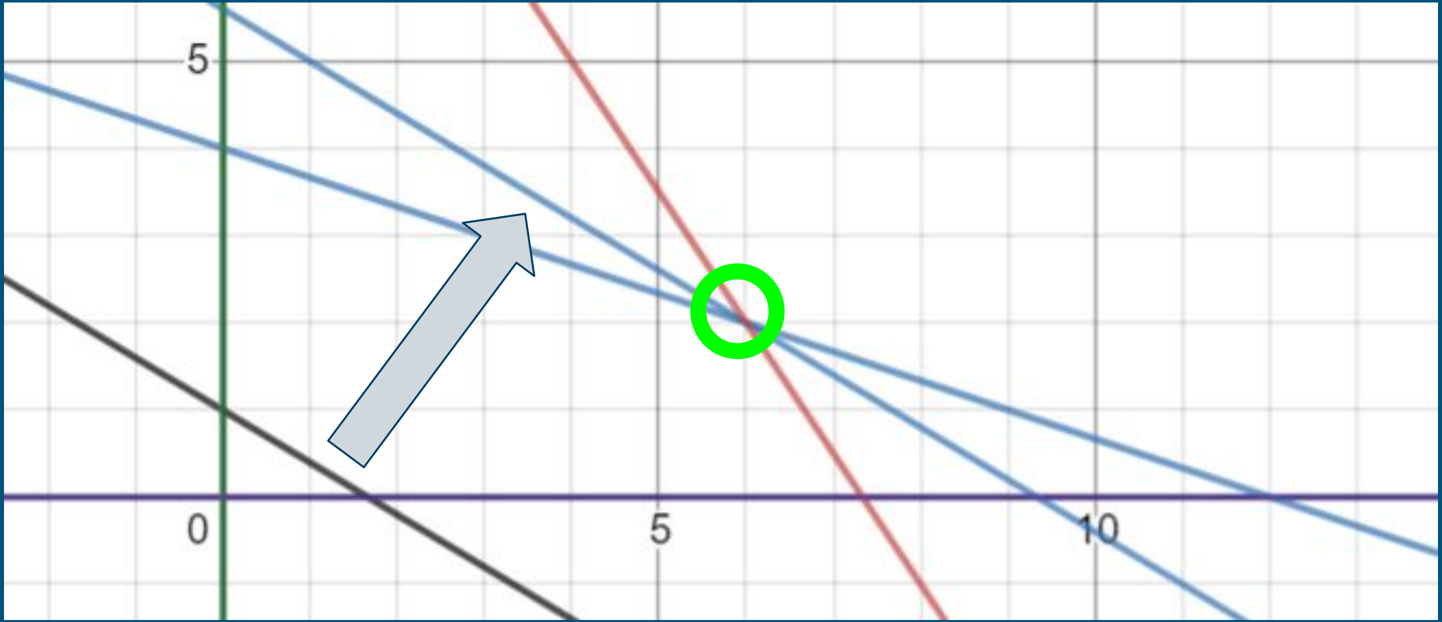


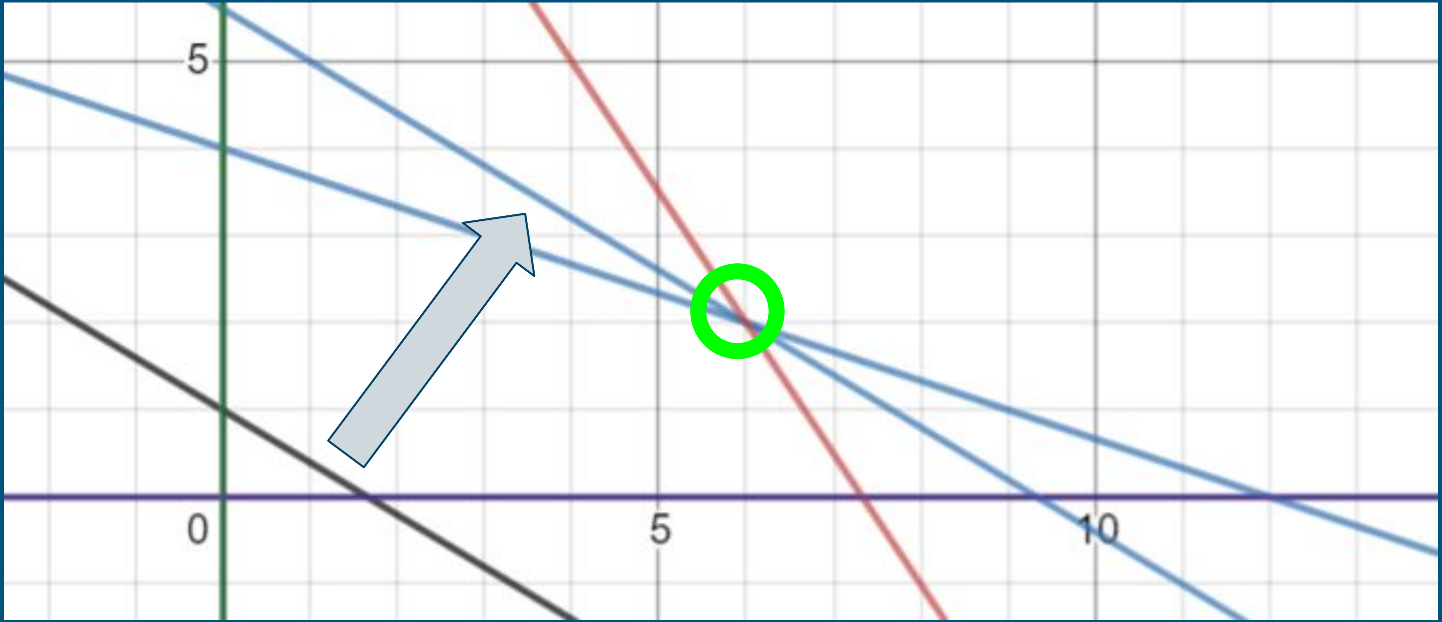
Preliminaries

Tangency point of feasible region and objective function.



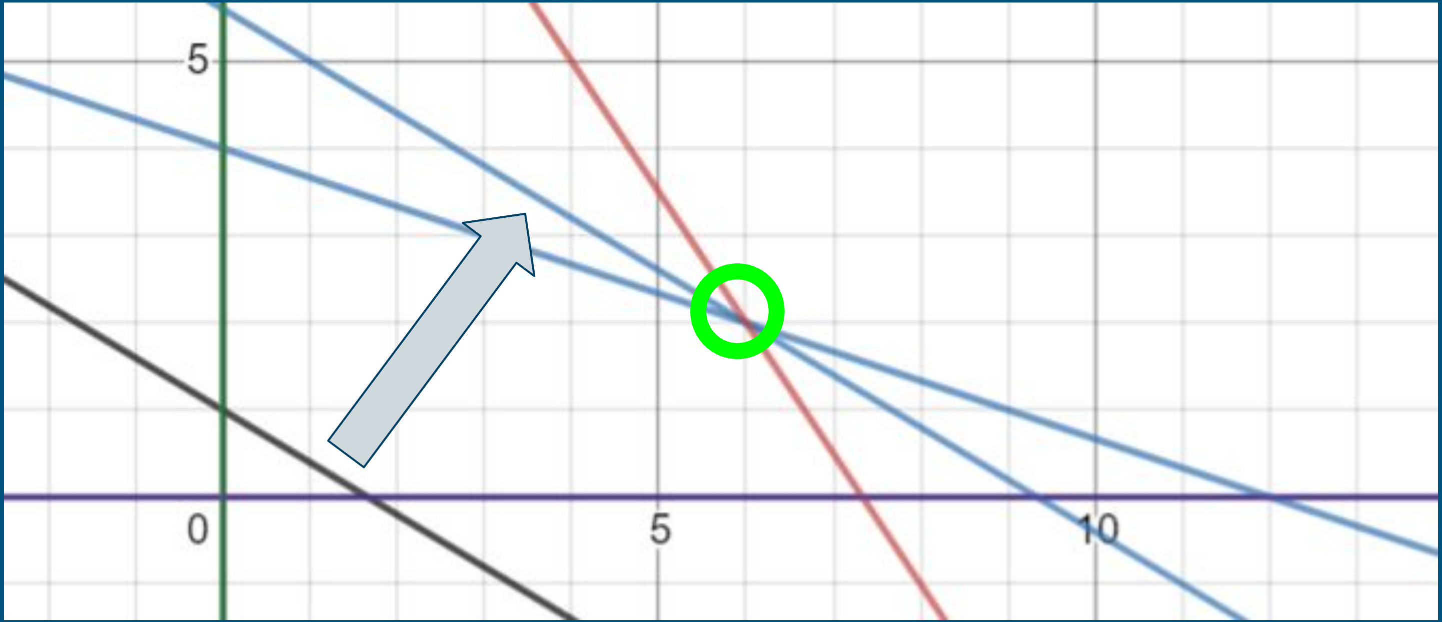
Preliminaries

This is the key idea behind the *simplex* algorithm.



Preliminaries

More on this later.



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   a. Yes! There are two ways this can happen:
      i. Unboundedness: Underconstrained, objective function goes to (negative) infinity.
      ii. Infeasibility: Overconstrained, no feasible region!

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4. How about equality constraints?
    a. Yes, introduce *slack* variables. Let constraints be $ax + by = c$, where $x, y \geq 0$. This is equivalent to: $ax + by - z \leq c$ where $x, y, z \geq 0$.

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Simplex algorithm:

1. Uses fact that the optimal point occurs where the constraints are *tight*.
2. Traverses across points of intersections between hyperplanes defined by the constraints.
3. Since there are an exponential number of these points, simplex algorithm is exponential.
4. However, simplex is extremely fast in practice!

Algorithms

There exist other algorithms for linear (convex) programs:

1. Ellipsoid method (polynomial time!... but big constants)
2. Interior point methods
3. Newton's method
4. Gradient/Conjugate Gradient Descent
5. Coordinate descent
6. Many others...

Duality

Important concept for linear programming. Original problem is called the *primal* problem. The *dual* problem is as follows:

Primal		Dual
Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge 0$	→	Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \ge \mathbf{c}$, $\mathbf{y} \ge 0$

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Let's formulate the dual for the example we had earlier:

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Recall our example from earlier:

Let x and y be the number of rings and necklaces to produce.

Maximize: $300x + 500y$

Subject to: $3x + 2y \le 22,$

$x + 3y \le 12,$

$x, y \ge 0$

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Things to note:

1. Objective function has the same value as the primal! There is a reason for this, which we will get into shortly.
2. The value of the objective function is the *minimum* amount that you would be willing to sell your inputs for; x' = 400/7$ per gold unit, y' = 900/7$ per diamond unit.

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Weak duality: For any feasible solution (x’, y’) in the dual and (x, y) in the primal, the objective function in the dual is at least that of the primal. (See proof in wiki for Dual Linear Program)

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Strong duality: If one of the two problems has an optimal solution, then the optimal objective function values are equal.

Extensions

Unfortunately, not all optimization problems are linear:

1. Quadratic programming	(exists nice theory for this)
2. Convex programming	(still somewhat nice theory)
3. Nonlinear optimization	(yikes)
4. Combinatorial optimization	(y i k e s)

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Can you think of an example?