Applications of Myerson's Lemma

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We apply Myerson's lemma to solve the single-good auction, and the generalization in which there are k homogeneous goods: i.e., k identical copies of the good. Our objective is welfare maximization.

1 *Welfare-Maximizing Auctions*

Myerson's lemma gives us a recipe for designing an IC and IR welfare-maximizing auction. The first step is to construct an allocation function that is monotonic in values, and the second step is to plug that function into the payment formula. When that monotonic allocation function also achieves economic efficiency (i.e., it optimizes, or approximately optimizes, welfare), and is also computationally efficient, we say that the auction is solved (or approximately solved).

2 *Single-Good Auction*

Our first application of Myerson's lemma is a simple sanity check. We have already discussed a DSIC auction design for the single-parameter setting with one good: the second-price auction, in which the highest bidder wins and pays the second-highest bid. Here, we comfirm that Myerson's lemma leads us to the same conclusion.

Welfare Maximization Recall that welfare is the quantity $\sum_i v_i x_i(\mathbf{v})$, where $\mathbf{x} \in \{0,1\}^n$ and $||\mathbf{x}|| \leq 1$. This quantity is maximized by awarding the good to a bidder with the highest value: i.e., a bidder i^* s.t.

$$i^* \in \arg\max_i v_i,$$

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. The necessary and sufficient condition for i to be allocated is that i bid higher than b^*, the **critical bid**, which is the highest bid among bidders other than i: i.e.,

$$b^* \equiv \max_{j \neq i} v_j,$$

This allocation rule is plotted in Figure 1.

Proposition 2.1. *This allocation rule is monotonically non-decreasing.*

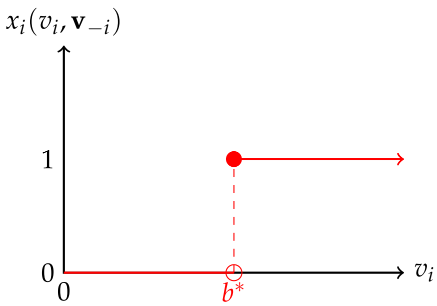

Figure 1: Bidder i's allocation rule, for a fixed $\mathbf{v}_{-i}$.

Proof. If $b_i < b^*$, then $x_i(b_i, \mathbf{v}_{-i}) = 0$, so increasing the bid cannot possibly lower the allocation. Indeed, for all $\epsilon > 0$, $x_i(b_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(b_i, \mathbf{v}_{-i})$. On the other hand, if $b_i \geq b^*$ is a winning bid, so that $x_i(b_i, \mathbf{v}_{-i}) = 1$, then for all $\epsilon > 0$, $x_i(b_i + \epsilon, \mathbf{v}_{-i})$ still equals 1. In particular, $x_i(b_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(b_i, \mathbf{v}_{-i})$. □

Payments By the payment formula, if $x_i = 0$, then $p_i = 0$. Therefore, only the winner of the auction will make a payment to the auctioneer. Assuming bidder i is a winner, their payment is as follows:

$$\begin{aligned}
p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z, \\
&= v_i \cdot 1 - \left[\int_0^{b^*} 0\,\mathrm{d}z + \int_{b^*}^{v_i} 1\,\mathrm{d}z\right] \\
&= v_i - (v_i - b^*) \\
&= b^*.
\end{aligned}$$

We split up the integral in this way because the allocation for bidding less than b^* bid is 0, while the allocation for bidding more is 1. This payment is the shaded region in Figure 2.

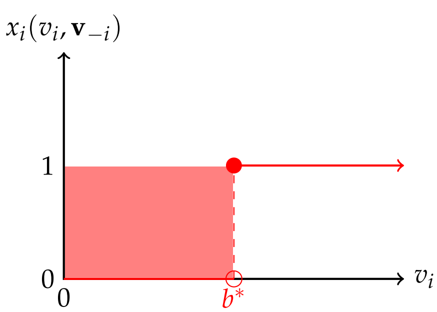

Figure 2: Bidder i's payment function, for a fixed $\mathbf{v}_{-i}$.

We conclude that the combination of an allocating to a highest bidder together with charging the winner of the auction the second-highest bid is IC and IR. Since this allocation rule is economically and compuationally efficient, this auction is solved.

3 *k-Good Auction*

In this auction, there are $k \geq 1$ identical copies of a good and $n \geq k$ bidders, each with a private value v_i for exactly one copy of the good (i.e., this is another single-parameter auction).

Welfare Maximization Problem Generalizing the single-good case, welfare is the quantity $\sum_i v_i x_i(\mathbf{v})$, where $\mathbf{x} \in \{0,1\}^n$ and $||\mathbf{x}|| \leq k$. This quantity is maximized by awarding the goods to the k highest bidders: i.e., by setting precisely those entries of $\mathbf{x}$ that correspond to the k largest bids to 1, and all others to 0.

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. The necessary and sufficient condition for i to be allocated is that i bid higher than b^*, the **critical bid**, which is the kth-highest bid among bidders other than i: i.e.,

$$b^* \equiv \max_{j \neq i} v_j,$$

Since the condition for being allocated is the same as it was in the single-good case, the allocation rule is the same as it was in the single-good case. This allocation rule is plotted in Figure 3.

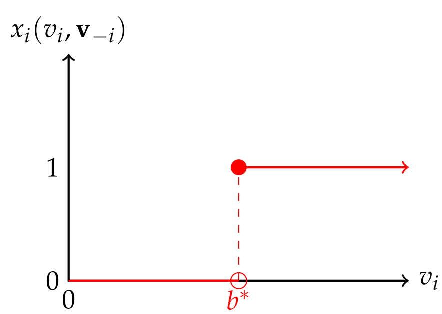

Figure 3: Bidder i's allocation rule, for a fixed $\mathbf{v}_{-i}$.

Proposition 3.1. *This allocation rule is monotonically non-decreasing.*

Proof. If $b_i < b^*$, then $x_i(b_i, \mathbf{v}_{-i}) = 0$, so increasing the bid cannot possibly lower the allocation. Indeed, for all $\epsilon > 0$, $x_i(b_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(b_i, \mathbf{v}_{-i})$. On the other hand, if $b_i \geq b^*$ is a winning bid, so that $x_i(b_i, \mathbf{v}_{-i}) = 1$, then for all $\epsilon > 0$, $x_i(b_i + \epsilon, \mathbf{v}_{-i})$ still equals 1. In particular, $x_i(b_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(b_i, \mathbf{v}_{-i})$. $\square$

Payments By the payment formula, if $x_i = 0$, then $p_i = 0$. Therefore, only the winners of the auction make a payment to the auctioneer. Assuming bidder i is a winner, their payment is as follows:

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z,$$

$$
\begin{aligned}
&= v_i \cdot 1 - \left[\int_0^{b^*} 0 \, \mathrm{d}z + \int_{b^*}^{v_i} 1 \, \mathrm{d}z \right] \\
&= v_i - (v_i - b^*) \\
&= b^*.
\end{aligned}
$$

Since the condition for being allocated is the same as it was in the single-good case—simply bidding higher than b^*—this payment calculation is the same as it was in the single-good case. This payment is the shaded region in Figure 4.

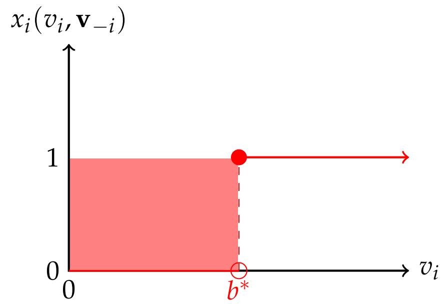

Figure 4: Bidder i's payment function, for a fixed $\mathbf{v}_{-i}$.

We conclude that the combination of allocating to the k highest bidders together with charging the winners of the auction the kth-highest bid is IC and IR. Since this allocation rule is economically and compuationally efficient, this auction is solved. This solution is called the k-Vickrey auction.

A two-good example. Imagine three bidders, b_1, b_2 and b_3, and two goods. The bidders' values are uniformly distributed on closed intervals, but with different bounds: each bidder i's value is uniformly distributed on the closed interval $[0, i]$, so $f_i(v) = \frac{1}{i}$, and $F_i(v) = \frac{v}{i}$, for all $v \in [0, i]$. Let v_i represent bidder i's realized value. Suppose $v_1 = 5/6$, $v_2 = 2$, and $v_3 = 7/4$. What happens in this example in the welfare-maximizing auction, IC, IR, and ex-post feasible auction?

To answer this question, we do the following:

1. Sort the bidders' values.

2. Find the winners: i.e., the bidders with the two highest values.

3. Determine the critical value, and hence the winners' payments.

These steps are illustrated in Table 1. Bidders 2 and 3 are allocated the goods, because they have the two highest values. They each pay the critical value, which in this example is the third-highest value.

i	v_i	Rank	Winner?	Critical bid	Payment
1	$5/6$	3	no	n/a	n/a
2	2	1	yes	$5/6$	$5/6$
3	$7/4$	2	yes	$5/6$	$5/6$

Table 1: Example Two-Good Auction