An Application of Myerson's Theorem

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We apply Myerson's theorem to solve the k-good auction, where there are k identical copies of a good. Our objective is revenue maximization.

1 *Revenue-Maximizing Auctions*

Myerson's theorem equates virtual welfare with revenue, so together with Myerson's lemma, we have a recipe for designing IC and IR revenue-maximizing auctions, assuming regularity. The first step is to construct an allocation function that is monotonic in virtual values, and the second step is to plug that function into the payment formula. When that monotonic allocation function also achieves economic efficiency (i.e., it optimizes, or approximately optimizes, virtual welfare, and hence revenue), and is also computationally efficient, we say that the auction is solved (or approximately solved).

2 *k-Good Auction*

Assume there are $k \geq 1$ identical copies of a good and $n \geq k$ bidders, each with a private value v_i for exactly one copy of the good.

Revenue Maximization By Myerson's theorem, to maximize revenue it suffices to maximize virtual welfare. We proceed as follows:

- Sort bidders by *virtual* value, so that $\varphi_1(v_1) \leq \varphi_2(v_2) \leq \cdots \leq \varphi_{n-1}(v_{n-1}) \leq \varphi_n(v_n)$.
- Allocate nothing to any bidders with negative virtual values, and to the lowest $n-k$ bidders.
- Among the remaining top $m \leq n$ bidders with non-negative virtual values, assign bidder $m+j-1$ good $k-j+1$, for $1 \leq j \leq k$.
  - m gets good k
  - $m+1$ gets good $k-1$
  - $m+2$ gets good $k-2$
  - $\vdots$
  - $m+k-1$ gets good 1

Remark 2.1. Although $n \geq k$, it is nonetheless possible that $m < k$ (when fewer than k bidders have non-negative virtual values). In this case, only m goods are assigned.

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. Define φ^* as the kth-highest virtual value among bidders other than i:

$$\varphi^* \equiv \max_{j \in N \setminus \{i\}} \varphi_j$$

Assuming $\varphi^* \geq 0$, the necessary and sufficient condition for i to be allocated is that they bid b s.t.

$$\varphi_i(b) \geq \varphi^*;$$

equivalently,

$$b \geq \varphi_i^{-1}(\varphi^*).$$

If $\varphi^* < 0$, then bidder i need not outbid anyone; they need only bid enough so that their own virtual value is non-negative: i.e.,

$$\varphi_i(b) \geq 0;$$

equivalently,

$$b \geq \varphi_i^{-1}(0).$$

The value $\varphi_i^{-1}(0)$ is called bidder i's **reserve** price.

This allocation rule can be summarized as follows: for $b \in T$,

$$x_i(b, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } \varphi_i(b) \geq 0 > \varphi^* \\ 1 & \text{if } \varphi_i(b) > \varphi^* \geq 0 \\ ? & \text{if } \varphi_i(b) = \varphi^* \geq 0 \\ 0 & \text{if } \varphi_i(b) < \varphi^* \end{cases} \tag{1}$$

That is, i is the sole winner if $\varphi_i(b) \geq 0$ and $\varphi_i(b) > \varphi^*$, while i is a loser if $\varphi_i(b) < \varphi^*$. On the other hand, if $\varphi_i(b) \geq 0$ and there is a tie for the highest virtual value, then the allocation is as-of-yet unspecified. We claim that this allocation rule is monotonic, if it is implemented with a deterministic tie-breaking rule.

Proposition 2.2. *If the tie-breaking rule is deterministic, this allocation rule (Equation 1) is monotonically non-decreasing.*

Proof. If $\varphi_i(b) < \max_{j \in N \setminus \{i\}} \varphi_j$, then $x_i(b, \mathbf{v}_{-i}) = 0$, so increasing the bid cannot possibly lower the allocation. Indeed, for all $\epsilon > 0$, $x_i(b + \epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i})$. On the other hand, if b is a winning bid, so that $x_i(b, \mathbf{v}_{-i}) = 1$, it must be the case that

$$\varphi_i(b) \geq \max_{j \in N \setminus \{i\}} \varphi_j(v_j).$$

Further, by regularity,

$$\varphi_i(b + \epsilon) \geq \max_{j \in N \setminus \{i\}} \varphi_j(v_j), \quad \forall \epsilon \geq 0, b + \epsilon \in T_i.$$

If this latter inequality is strict, then $x_i(b+\epsilon, \mathbf{v}_{-i}) = 1$, so that for all $\epsilon > 0$, $x_i(b+\epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i})$.

If this latter inequality is not strict, then there are ties among virtual values. Likewise, there must have been ties among virtual values at bid b. But since $x_i(b, \mathbf{v}_{-i}) = 1$, any such ties were broken in i's favor at bid b. Further, since the tie-breaking rule is deterministic, any such ties would again be broken in i's favor at bid $b+\epsilon$. Therefore, for all $\epsilon > 0$, $x_i(b+\epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i})$. □

Generally speaking (i.e., for an arbitrary tie-breaking rule), it is possible that ties are broken in i's favor at bid b, but otherwise at bid $b+\epsilon$: i.e., $x_i(b, \mathbf{v}_{-i}) = 1$, while $x_i(b+\epsilon, \mathbf{v}_{-i}) = 0$. However, taking expectations over ex-post allocations w.r.t. any randomness in the mechanism yields a deterministic allocation rule, which by Proposition 2.2 is indeed monotonically non-decreasing. Therefore:

Proposition 2.3. *Ex-ante, this allocation rule (Equation 1) is monotonically non-decreasing.*

Payments By the payment formula, if $x_i = 0$, then $p_i = 0$. Therefore, only the winners of the auction make a payment to the auctioneer. Letting b^* denote the critical value above which bidder i is allocated, bidder i's payment is as follows:

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, dz, \\ &= v_i \cdot 1 - \left[\int_0^{b^*} 0\, dz + \int_{b^*}^{v_i} 1\, dz\right] \\ &= v_i - (v_i - b^*) \\ &= b^*. \end{aligned}$$

Payments in the revenue-maximizing auction are syntactically equivalent to the payments in the welfare-maximizing auction. They differ, however, in the meaning of b^*. Whereas b^* was the $k+1$st-highest bid in the welfare-maximizing case, here it is the inverse, according to i's virtual value function, of φ^* (the kth-highest virtual value among bidders other than i), assuming φ^* is non-negative; or, if φ^* is negative, it is the inverse, according to i's virtual value function, of 0. In sum, payments are given by:

$$p_i(v_i, \mathbf{v}_{-i}) = \begin{cases} \varphi_i^{-1}(\varphi^*) & \text{if } \varphi^* \geq 0 \\ \varphi_i^{-1}(0) & \text{otherwise} \end{cases}$$

N.B. $\varphi_i^{-1}(\varphi^*) \geq \varphi_i^{-1}(0)$ whenever $\varphi^* \geq 0$, by the regularity assumption (i.e., the virtual value function is non-decreasing).

3 *A Revenue-Maximizing Two-Good Auction*

Imagine three bidders, b_1, b_2 and b_3, and two goods. The bidders' values are uniformly distributed on closed intervals, but with different bounds: each bidder i's value is uniformly distributed on the closed interval $[0, i]$, so $f_i(v) = \frac{1}{i}$, and $F_i(v) = \frac{v}{i}$, for all $v \in [0, i]$. Let v_i represent bidder i's realized value. Suppose $v_1 = 5/6$, $v_2 = 2$, and $v_3 = 7/4$. What happens in this example in the revenue-maximizing auction, IC, IR, and ex-post feasible auction?

To answer this question, we do the following:

1. Calculate the virtual value function for each bidder.
2. Find each bidder's virtual value.
3. Sort the virtual values.
4. Throw out the bidders with negative virtual values.
5. Among the remaining bidders, find the winners: i.e., the bidders with the highest virtual value.
6. Determine the critical value, and hence each winner's payment.

Table 1: Example Two-Good Auction

i	v_i	$F_i(v)$	$\varphi_i(v)$	$\varphi_i(v_i)$	RANK	$\varphi_i(v_i) \geq 0$?	WINNER?	CRITICAL BID	PAYMENT
1	$5/6$	v	$2v-1$	$2/3$	2	YES	YES	$\varphi_1^{-1}(1/2)$	$3/4$
2	2	$v/2$	$2v-2$	2	1	NO	YES	$\varphi_2^{-1}(1/2)$	$5/4$
3	$7/4$	$v/3$	$2v-3$	$1/2$	3	NO	NO	N/A	N/A

These steps are illustrated in Table 1. Bidders 1 and 2 are allocated the goods, because they have the two highest virtual values, and neither of their virtual values are negative. They each pay the inverse of their virtual value function at the critical value, which in this example is the third-highest virtual value, since that value is not negative. In Table 2, the third-highest virtual value *is* negative, so the winning bidders pay the inverse of their virtual value function at 0.

Table 2: Example Two-Good Auction

i	v_i	$F_i(v)$	$\varphi_i(v)$	$\varphi_i(v_i)$	RANK	$\varphi_i(v_i) \geq 0$?	WINNER?	CRITICAL BID	PAYMENT
1	$5/6$	v	$2v-1$	$2/3$	2	YES	YES	$\varphi_1^{-1}(0)$	$1/2$
2	2	$v/2$	$2v-2$	2	1	YES	YES	$\varphi_2^{-1}(0)$	1
3	1	$v/3$	$2v-3$	-1	3	NO	NO	N/A	N/A