Bayes-Nash Equilibrium in the First-Price Auction

CS 1951k/2951z

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We state and prove a symmetric Bayes-Nash Equilibrium strategy for the first-price auction, assuming IPV, and that values are drawn i.i.d. from the uniform distribution on $[0,1]$.

Theorem 0.1. *In a first-price auction, assuming IPV and that values are drawn i.i.d. from the uniform distribution on* $[0,1]$*, the bidding strategy* $b_i = \left(\frac{n-1}{n}\right) v_i$ *comprises a symmetric Bayes-Nash equilibrium.*

Proof. Fix a bidder i. We assume that all bidders besides i bid according to this formula, and argue that bidder i should do the same.

Let v' denote the highest bid among the $n-1$ bidders other than i. Let z represent i's bid.

Now there are two possible cases:

- $z < \left(\frac{n-1}{n}\right) v'$. In this case, i does not win the item, so $u_i = 0$.
- $z \geq \left(\frac{n-1}{n}\right) v'$. In this case, i wins the item, so $u_i = v_i - z$. The probability of this event is computed as follows:

$$\begin{aligned}
\Pr\left(z \geq \left(\frac{n-1}{n}\right) v'\right) &= \Pr\left(v' \leq \frac{nz}{n-1}\right) \\
&= \Pr\left(v_j \leq \frac{nz}{n-1}, \forall j\right) \\
&= \prod_{j \neq i} F_{v_j \sim \mathrm{Unif}[0,1]} \left(\frac{nz}{n-1}\right) \\
&= \prod_{j \neq i} \left(\frac{nz}{n-1}\right) \\
&= \left(\frac{nz}{n-1}\right)^{n-1}
\end{aligned}$$

Technical Note. If $z > \frac{n-1}{n}$, then $\left(\frac{nz}{n-1}\right)^{n-1} > 1$, and the cdf is 1:

$$\begin{cases} \left(\frac{nz}{n-1}\right)^{n-1} & \text{if } z \leq \frac{n-1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Since, a bid of $\frac{n-1}{n}$ guarantees that i wins the auction, we restrict the value of z to the range $[0, \frac{n-1}{n}]$. $\diamond$

When bidding $z \in [0, \frac{n-1}{n}]$, bidder i's expected utility is given by:

$$\mathbb{E}[u_i] = \left(\frac{nz}{n-1}\right)^{n-1} (v_i - z) + \left(1 - \left(\frac{nz}{n-1}\right)^{n-1}\right) \cdot 0$$

$$= \left(\frac{n}{n-1}\right)^{n-1} z^{n-1}(v_i - z).$$

Next, we take the derivative of $\mathbb{E}[u_i]$ with respect to z, and set it equal to 0, to maximize i's expected utility. Since $\left(\frac{n}{n-1}\right)^{n-1}$ is a just a constant, it eventually drops out, so we drop it from the start:

$$\frac{d\mathbb{E}[u_i]}{dz} = \frac{d}{dz}\left[z^{n-1}(v_i - z)\right] = (n-1)z^{n-2}(v_i - z) - z^{n-1}$$

Setting this derative equal to zero:

$$\begin{aligned} \frac{d\mathbb{E}[u_i]}{dz} &= 0 \\ (n-1)z^{n-2}(v_i - z) - z^{n-1} &= 0 \\ (n-1)(v_i - z) - z &= 0 \\ (n-1)v_i - nz &= 0 \end{aligned}$$

Therefore, bidder i maximizes her utility by bidding:

$$z = \left(\frac{n-1}{n}\right) v_i.$$

To be sure of this, we also must verify that neither of the two extreme points $z = 0$ and $z = \frac{n-1}{n}$ yields positive utility, and that the second derivative of $\mathbb{E}[u_i](z)$ is negative at $z = \left(\frac{n-1}{n}\right) v_i$. We leave these remaining steps as an exercise for the reader. □