Myerson's Lemma with Discrete Values

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Myerson's lemma, as presented, uses continuous variables; regardless, his results continue to hold when values are discretized.

1 *Myerson with Discrete Values*

Let $z_{i,1}, \ldots, z_{i,M_i}$ be the M_i discretized values of bidder i, where $z_{i,j+1} \geq z_{i,j}$, for all $1 \leq j \leq M_i - 1$, and let $z_{i,M_i+1} = z_{i,M_i}$.

In the discrete setting, incentive compatibility and individual rationality take the following form:

$$z_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i}) - p_i(z_{i,k}, \mathbf{v}_{-i}) \geq z_{i,k} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - p_i(z_{i,\ell}, \mathbf{v}_{-i}), \qquad \forall i \in N, \forall z_{i,k}, z_{i,\ell} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i} \tag{1}$$

$$z_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i}) - p_i(z_{i,k}, \mathbf{v}_{-i}) \geq 0, \qquad \forall i \in N, \forall z_{i,k}, z_{i,\ell} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \tag{2}$$

Myerson's lemma also holds in the discrete setting, so that incentive compatibility and individual rationality can be replaced with a monotonic allocation rule and a corresponding payment rule. In the discrete setting, however, the payment rule is not unique! One possible payment rule is the following:

$$p_i(z_{i,\ell}, \mathbf{v}_{-i}) = z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} \left(z_{i,j+1} - z_{i,j}\right) x_i\left(z_{i,j}, \mathbf{v}_{-i}\right), \quad \forall i \in N, \forall z_{i,\ell} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \tag{3}$$

If one uses this equation as the payment formula, then the virtual value, $\psi_i(z_{i,k})$, of the kth type, $z_{i,k}$, is given by:

$$\psi_i(z_{i,k}) \equiv \psi_i(z_{i,k}, z_{i,k+1}) = z_{i,k} - (z_{i,k+1} - z_{i,k}) \left(\frac{1 - F_i(z_{i,k})}{f_i(z_{i,k})} \right). \tag{4}$$

2 *Welfare-Maximization with Discrete Values*

The discrete analog of the pointwise welfare-maximization algorithm is given in Algorithm 1. As in the continuous case, allocations are solved for first (using the greedy algorithm, which is monotone), based on which appropriate payments are constructed.

Algorithm 1: Welfare Maximization. This algorithm proceeds **pointwise**, one valuation vector at a time.

for all $\mathbf{v} \in T$ **do** ▷ Find allocations
  for all $i \in N$ **do**
    $x_i(v_i, \mathbf{v}_{-i}) \leftarrow 0$
  end for
  if $\max_i\{v_i : i \in N\} > 0$ **then**
    $w(\mathbf{v}) \leftarrow \arg\max_i\{v_i : i \in N\}$
    for all $i^* \in w(\mathbf{v})$ **do**
      $x_{i^*}(v_{i^*}, \mathbf{v}_{-i^*}) \leftarrow 1/|w(\mathbf{v})|$
    end for
  end if
end for
for all $i \in N$ **do** ▷ Compute payments
  for all $\mathbf{v} \in T$ **do**
    Find ℓ such that $z_{i,\ell} = v_i$
    $p_i(z_{i,\ell}, \mathbf{v}_{-i}) \leftarrow z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} \left(z_{i,j+1} - z_{i,j}\right) x_i\left(z_{i,j}, \mathbf{v}_{-i}\right)$
  end for
end for
$S \leftarrow \sum_{i=1}^{n} \mathbb{E}_{\mathbf{v}\sim F}\left[v_i x_i(v_i, \mathbf{v}_{-i})\right]$ ▷ Total expected welfare
$R \leftarrow \sum_{i=1}^{n} \mathbb{E}_{\mathbf{v}\sim F}\left[p_i(v_i, \mathbf{v}_{-i})\right]$ ▷ Total expected revenue
return S, R, $\mathbf{x}$, $\mathbf{p}$