# The Clinching Auction CSCI 1951k/2951z

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We introduce a new multi-parameter setting, for which we can find an approximately welfare-maximizing EPIC auction. We prove the EPIC property by making use of our recipe for doing so: 1. we prove sincere bidding in the auction yields an approximate VCG outcome, and 2. we show consistent bidding strategies dominate inconsistent ones. Not only does this mechanism satisfy desired performance and incentive guarantees (up to some additive error), it is also tractable.

# 1 Diminishing Marginal Valuations

We introduce a new instance of the multi-parameter setting, so-called **diminishing marginal valuations** for homogeneous goods. Rather than additive or unit-demand valuations, here each bidder's marginal value for an additional copy of the good is non-increasing. Formally,

- We assume *n* bidders and *m* identical goods, with bidders indexed by *i*, and goods, by *j*.
- Each bidder *i* has a marginal value µ<sub>i</sub>(*j*) for its *j*th copy of the good, meaning its value for acquiring a *j*th copy of the good, given they already have *j* 1 copies in its possession.
- Each bidder *i*'s marginal values are non-increasing: µ<sub>i</sub>(1) ≥ µ<sub>i</sub>(2) ≥ · · · ≥ µ<sub>i</sub>(m).

Our goal is to construct an approximately welfare-maximizing EPIC multi-unit ascending auction for this scenario.

# 2 A Direct Mechanism: A Sanity Check

By the revelation principle, a welfare-maximizing DSIC direct mechanism "reduces to"<sup>1</sup> a welfare-maximizing EPIC indirect mechanism, in the sense that a polynomial-time solution to the latter can be used to construct a polynomial-time solution to the former. Therefore, solving for an EPIC indirect mechanism with all our desiderata is at least as hard as solving for a DSIC direct mechanism with the same desiderata. In other words, if no such DSIC direct mechanism exists (one that is welfare-maximizing in polynomial time), neither can such an EPIC indirect mechanism.

As a result of this argument, before we embark upon the design of an EPIC indirect mechanism that maximizes welfare in polynomial <sup>1</sup> See Appendix A

time, we best do a quick sanity check: can we design a DSIC direct mechanism that maximizes welfare in polynomial time?

Fortunately, we can solve this problem in the affirmative in the direct setting. In particular, the welfare-maximizing allocation can be computed via a simple greedy allocation algorithm:

- Collect a vector of bids b from all bidders *i* ∈ *N*, with bid *b<sub>i</sub>(j)* representing *i*'s bid on the *j*th copy of the good.
- Sort the bids, and then allocate the goods to the bidders who submitted the highest *m* bids, breaking ties arbitrarily.

For example, if  $b_i(4)$  is among the highest *m* bids, but  $b_i(5)$  is not, then bidder *i* is allocated four goods, which we denote as  $x_i = 4$ .

As usual, to achieve the VCG outcome, we combine this allocation algorithm with payments that charge bidders their externalities. For each bidder *i*, we sort the bids by *bidders other than bidder i* from greatest to least, and then establish the following groupings.

$$\underbrace{\beta_1 \quad \beta_2 \quad \cdots \quad \beta_{m-x_i}}_{A} \quad \underbrace{\beta_{m-x_i+1} \quad \beta_{m-x_i+2} \quad \cdots \quad \beta_m}_{B} \quad \underbrace{\beta_{m+1} \quad \beta_{m+2} \quad \cdots \quad \beta_{mn}}_{C}$$

The bids in group A are those that are allocated regardless of i's presence. The bids in group C are those that are *not* allocated regardless of i's presence. The bids in group B are those whose allocation depends on i's presence. These bids comprise bidder i's externality. We therefore charge bidder i, in total, for all  $x_i$  goods in group B, the sum of these  $x_i$  bids: i.e.,

$$p_i(x_i) = \sum_{j=1}^{x_i} \beta_{m-x_i+j}$$

By charging each bidder its externality, we charge the VCG payments, thereby guaranteeing the DSIC property.

Once again, when bidder *i* is allocated  $x_i$  copies of the good, its VCG payment is the sum of  $x_i$  bids, one per copy of the good. We can interpret these bids as follows: the smallest bid is bidder *i*'s payment for its first copy of the good; the second-smallest bid is its payment for its second copy of the good; and so on. Note that payments for additional copies are *non-decreasing*, although values are (by assumption) non-increasing. Payments are non-decreasing because the first copy allocated to bidder *i* displaces only bid  $\beta_m$ , whereas the second copy displaces bid  $\beta_{m-1} \ge \beta_m$ , and so on.

Building on these observations, we can express bidder *i*'s VCG payment for good *j* in terms of the other bidders' demand sets. Define bidder *k*'s **demand set** at price *q*,  $D_k(q) = \max_j \{ j \le m \mid v_k(j) \ge q \}$ .

Now, the price of bidder *i*'s *j*th copy is given by:

$$p_i(j) = \inf\left\{ q \left| \sum_{k \neq i} D_k(q) \le m - j \right\}.$$
(1)

This price is the price at which the demand of all other bidders falls below m - j. As expected, these prices are again non-decreasing. Each additional copy of the good costs *no less* than the previous, as other bidders' demands fall as the price rises.

#### 3 An Indirect Mechanism: The Clinching Auction

Having satisfied the precondition for potential success, we now set our sights on an EPIC welfare-maximizing ascending auction. We present the following auction, called the **clinching auction**:<sup>2</sup>

- Initialize q = 0.
- Collect demand sets from all bidders. (Initially, when q = 0, it should be that D<sub>i</sub>(q) = m, for all bidders *i*.)
- Alternate between incrementing *q* by *ε* and collecting demand sets from bidders until ∑<sup>n</sup><sub>i=1</sub> D<sub>i</sub>(q) ≤ m.
- Activity rule: Ensure that no bidder's demand increases over time: i.e., bidder's demands can only decrease as prices increase.
- Allocate to bidder *i* its final demand, namely D<sub>i</sub>(q) goods. If unallocated goods remain, allocate them randomly to bidders *i* with leftover demand at price q − ε: i.e., bidders *i* for whom D<sub>i</sub>(q − ε) − D<sub>i</sub>(q) > 0.
- Charge bidder *i* (within *e* of) its externality. Specifically, charge bidder *i* for its *j*th good:

$$q_i(j) = -\epsilon + \min_{t \in \mathbb{Z}^+} \left\{ \epsilon t \mid \sum_{k \neq i} D_k(\epsilon t) \le m - j \right\}.$$
(2)

As intended, this price is again (near) the price at which the demand of all other bidders falls below m - j.

**N.B.** When  $\sum_{k \neq i} D_k(\epsilon t) = m - j$ , it is not necessary to subtract  $\epsilon$  from  $\epsilon t$ . But when  $\sum_{k \neq i} D_k(\epsilon t) < m - j$ , the situation is analogous to the last remaining bidders dropping out at the same time in an English auction for one good, in which case the good is sold at the final price less  $\epsilon$  to ensure individual rationality.

<sup>2</sup> Lawrence M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94(5):1452–1475, December 2004 **Example 3.1.** Here is an example of a run-through of the clinching auction lifted from the paper that introduced it. The example is loosely based on the first US Nationwide Narrowband spectrum auction in where there were five bidders and five licenses, with the constraint that no bidder could win more than three licenses.

The bidders' marginal values are as follows:

License	А	В	С	D	Е
First	123	74	125	84	44
Second	113	5	125	64	24
Third	103	3	49	7	5

The auction then proceeds as follows, with demands depicted only at the most relevant prices:

Price	А	В	С	D	Е
10	3	1	3	2	2
25	3	1	3	2	1
45	3	1	3	2	0
50	3	1	2	2	0
65	3	1	2	1	0
75	3	0	2	1	0
85	3	0	2	0	0

At price 65, the demands of all bidders other than bidder A falls below the total supply of 5. Hence, bidder A "clinches" its first license at this price.<sup>3</sup> The license is said to be clinched because the fact that the other bidders' demands have fallen below 5 guarantees that bidder A will win this license.

At price 75, the demands of all bidders other than bidder A falls below 4, so A clinches its second license at this price. In addition, the demands of all bidders other than bidder C falls below 5, so C clinches its first license at this price. The auction terminates at price 85, when total demand meets total supply. At this price, bidder A clinches its third license, and bidder C, its second.

In sum, bidder A pays 65 + 75 + 85 for its three licenses, and bidder C pays 75 + 85 for its two licenses. As expected, prices on additional licenses are non-decreasing (strictly increasing, in fact).

The outcome of the clinching auction in this example (and always; see Proposition 4.1) is efficient (up to  $m\epsilon$ ). In contrast, in this example, a uniform-price auction<sup>4</sup> would *not* have yielded an efficient outcome, as it would have been in bidder *A*'s best interest to decrease its demand to two licenses when the price reached \$75 rather than

<sup>3</sup> Technically, the price should be 65 less  $\epsilon$ , but we ignore this adjustment factor in this example.

<sup>4</sup> A uniform-price auction is one that charges the same price for all copies of the good. win all three licenses for \$85 each. Winning only two licenses, A's utility would have been 236 - 150 = 86, whereas winning all three, A's utility would have been 339 - 255 = 84.

## 4 The Clinching Auction is EPIC

To prove the clinching auction is approximately EPIC, we follow the design recipe for EPIC auctions. That is, we first show that the outcome of sincere bidding in the clinching auction is approximately VCG (both allocation and payments; Steps 1 and 2 of the design recipe, respectively). We then show that no inconsistent deviations are preferable to sincere bidding (Step 3).

Regarding Steps 1 and 2, by design (Equations 1 and 2), each VCG price lower bounds the corresponding price in the clinching auction up to  $\epsilon$ . Thus, it sufficies to show the corresponding upper bound, and to show that any allocation arrived at via the clinching auction is welfare maximizing up to *m* $\epsilon$ . The proof of this latter claim (i.e., of the following proposition) is nearly identical to the proof that sincere bidding in an *m*-good English auction yields total welfare within *m* $\epsilon$  of the optimal, assuming unit-demand valuations.<sup>5</sup>

**Proposition 4.1.** Assuming sincere bidding, the clinching auction yields total welfare within  $m\epsilon$  of the optimal.

To show that VCG prices upper bound the prices in the clinching auction up to *me* assuming sincere bidding, we actually show that *utility* in a VCG auction *lower* bounds *utility* in the clinching auction assuming sincere bidding, up to *me*. If each bidder's utility in the former is not very different than its utility in the latter, then prices cannot be very different either. (We switch from proving an upper bound to proving a lower bound, because prices are negated in utility calculations.)

**Theorem 4.2.** An arbitrary bidder's utility in a VCG auction is bounded above by its utility in the clinching auction plus  $m\epsilon$ , assuming all bidders bid sincerely.

*Proof.* By Equation 1, the utility of bidder *i* in the VCG auction, assuming truthful bidding, is given by:

$$\sum_{j=1}^{x_i} (\mu_i(j) - p_i(j)).$$
(3)

Similarly, by Equation 2, the utility of bidder *i* in the clinching auction, assuming sincere bidding, is given by:

$$\sum_{j=1}^{g_i} \left( \mu_i(j) - q_i(j) \right).$$
 (4)

<sup>5</sup> Homework 8, Problem 3, Part 1

In these expressions,  $x_i$  ( $y_i$ ) is the number of copies of the good that bidder i wins in the VCG (clinching) auction. For all copies j of the good that bidder i wins in both auctions,  $\mu_i(j) - q_i(j) \ge \mu_i(j) - p_i(j)$ , because  $q_i(j) \le p_i(j)$  by definition. The proof thus concentrates on the cases where  $x_i$  and  $y_i$  differ,  $x_i < y_i$  and  $y_i < x_i$ . In particular, we show the following:

- To the extent that y<sub>i</sub> < x<sub>i</sub> (i.e., bidder *i* wins fewer copies in the clinching auction than in VCG), the value added by any term that appears in Equation 3 but not in Equation 4 is bounded above by *ε*, so that in total any missing terms (i.e., missing copies of the good) can forego at most mε > 0 utility for bidder *i*.
- To the extent that y<sub>i</sub> > x<sub>i</sub> (i.e., bidder *i* wins more copies in the clinching auction than in VCG), the value added by any term that appears in Equation 4 but not in Equation 3 is bounded below by o, so that in total any extra terms (i.e., extra copies of the good) always contribute at least 0 utility for bidder *i*.

**Upper Bound.** Consider a copy *j* of the good for which  $\mu_i(j) - p_i(j) > \epsilon$ . Then the price *q* at some point during the clinching auction is such that  $\mu_i(j) > q$  (i.e.,  $D_i(q) \ge j$ ) and  $\sum_{k \ne i} D_k(q) \le m - j$ . Therefore, *i* wins at least *j* copies of the good upon termination. Equivalently, as per the contrapositive, if *i* does not with the *j*th copy of the good, then  $\mu_i(j) - p_i(j) \le \epsilon$ .

**Lower Bound.** Let  $q^*$  denote the final price in the clinching auction. If *i* wins at least *j* copies of the good, then  $D_i(q^* - \epsilon) \ge j$ : i.e., *i* demanded at least *j* copies when the price was  $q^* - \epsilon$ . In other words,  $\mu_i(j) \ge q^* - \epsilon$ . Moreover,  $q^* - \epsilon \ge q_i(j)$ , because either j = m (*i* wins the last good), in which case  $q^* - \epsilon = q_i(j)$  (*i* pays the final price less  $\epsilon$ ), or j < m, in which case  $q^* - \epsilon > q_i(j)$ . Either way,  $\mu_i(j) \ge q_i(j)$ , so bidder *i*'s utility for each of the *j* copies it wins is non-negative.

Having completed Steps 1 and 2 of the EPIC auction design recipe, we have established that sincere bidding in the clinching auction is an EPNE up to  $m\epsilon$ , among consistent strategies. The remaining piece of this puzzle, then, is to further show that sincere bidding in the clinching auction is an EPNE up to  $m\epsilon$ , among consistent *and inconsistent* strategies: i.e., that no inconsistent deviations would yield substantially greater utility than sincere bidding. This claim is established in the following theorem.

**Theorem 4.3.** The clinching auction is EPIC, up to  $m\epsilon$ .

*Proof.* Assume all bidders except bidder *i* bid sincerely. Consequently, other bidders' behaviors are not impacted by *i*'s strategy.

Moreover, *i*'s payments are dictated entirely by the other bidders' demands, which again, *i* cannot influence.

Under these circumstances, we argue that *i* cannot benefit from bidding inconsistently. To bid inconsistently in the clinching auction would be to report false demand sets. But, *given the activity rule*,<sup>6</sup> which ensures that bidders' demands never increase, all such reports are in fact consistent with some valuation or another. So it is not actually possible to bid inconsistently in the clinching auction.

We have already established that bidding sincerely in the clinching auction is an EPNE up to  $m\epsilon$ , among consistent strategies. As there are no inconsistent strategies, it is likewise EPIC up to  $m\epsilon$ .

## A Reductions: A Primer

There are three steps in the reduction process, from a known hard problem O (for old) to a new problem N (for new), to show N is also hard (e.g., NP-hard).

- Pick a known hard problem *O*.
- Assume a polynomial-time algorithm for the new problem *N*.
- Derive a polynomial-time algorithm for *O* using the polynomial-time algorithm for *N* as a subroutine.

These three steps lead to a contradiction, as O is known to be hard. Therefore, there can be no polynomial-time algorithm for N (unless P = NP).

We apply these three steps to reduce the design of a multi-parameter DSIC direct auction (i.e., VCG) to the design of a multi-parameter EPIC indirect auction. We assume general valuations: i.e., we assume only monotonicity and free disposal.

- Pick designing a multi-parameter DSIC auction (i.e., computing a VCG outcome), a known hard problem.
- Assume a polynomial-time algorithm for the design of a multiparameter EPIC indirect auction.
- We can thus compute a VCG outcome in polynomial time by applying the revelation principle, because the outcome of the EPIC indirect auction, in which sincere bidding is an EPNE, would yield a DSIC direct auction, in which truthful bidding is a DSE (i.e., a VCG outcome).

These three steps lead to a contradiction, as designing a multiparameter DSIC direct auction (i.e., computing a VCG outcome) is <sup>6</sup> At long last, we discover the purpose of the activity rule. It rules out inconsistent bidding.

known to be NP-hard. Therefore, there can be no polynomial-time algorithm for the design of a multiparameter EPIC indirect auction (unless P = NP) in the general case.

In the special case of diminishing marginal valuations, or unitdemand valuations, where computing the VCG outcome is *not* NPhard, this argument merely implies that designing a multi-parameter EPIC indirect auction is at least as hard as (i.e., is no easier than) measured on a complexity theory scale—designing a multi-parameter DSIC direct auction, because *if there were a more efficient solution to the indirect auction design problem, it would just as well apply to the direct auction design problem via the revelation principle.* 

# References

 Lawrence M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94(5):1452–1475, December 2004.