Bayesian Battle of the Sexes

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We present an example of a Bayesian game. This set of notes is partially based on this video.

1 *Where Should Alice and Bob See a Movie Tonight?*

Two roommates, Alice and Bob, are planning to see a movie tonight, at one of two possible locations: the cinema (C), or at home (H). Alice is interested in Bob, and would like to be in the same place as Bob. However, we do not know if Bob is interested (I), or uninterested (U) in Alice.[1] If Bob is also interested in Alice, then he also receives positive payoff for being with Alice. Conversely, if Bob is not interested in Alice, then he receives zero payoff for begin with Alice. Each of Bob's types are equally likely (so $\Pr(I) = \Pr(U) = \frac{1}{2}$).

[1] And they don't communicate beforehand, because ... *reasons*.

If Bob is interested in Alice, the utility Alice and Bob receive are given by Figure 1, where Alice is the row player, and Bob is the column player.

	C	H
C	10, 5	0, 0
H	0, 0	5, 10

$\Pr(I) = \frac{1}{2}$

Figure 1: The payoff matrix describing the payoffs Alice and Bob receive for attending C or H, if Bob is interested in Alice. Alice is the row player.

On the other and, if Bob is not interested in Alice, the utility Alice and Bob receive are given by Figure 2, where Alice is again the row player, and Bob, the column player.

	C	H
C	10, 0	0, 10
H	0, 5	5, 0

$\Pr(U) = \frac{1}{2}$

Figure 2: The payoff matrix describing the payoffs Alice and Bob receive for attending C or H, if Bob is *not* interested in Alice. Alice is the row player.

2 *Representing this Bayesian Game in the Normal Form*

The ex-ante expected utility of player i assuming strategy profile $\mathbf{s}$ is:

$$\mathbb{E}\left[u_i(\mathbf{s})\right] = \mathop{\mathbb{E}}_{\mathbf{t}\sim F}\left[u_i(\mathbf{s};\mathbf{t})\right]. \tag{1}$$

Using this formula, we can describe ex-ante expected utilities for any strategy profile in our Bayesian game, which leads to a representation of the game in the normal form.

For example, suppose Alice plays C, and Bob plays C if he has type I, and H if he has type U. (We use the notation CH as shorthand to describe Bob's strategy.) Then, Alice's expected utility is:

$$\begin{aligned}
\mathbb{E}\left[u_a(C, CH)\right] &= \sum_{\mathbf{t} \in T} \Pr(\mathbf{t})\, u_a(C(t_a), CH(t_b); \mathbf{t}) && (2)\\
&= \Pr(I)\, u_a(C(\cdot), CH(I); (\cdot, I)) + \Pr(U)\, u_a(C(\cdot), CH(U); (\cdot, U)) && (3)\\
&= \Pr(I)\, u_a(C, C; (\cdot, I)) + \Pr(U)\, u_a(C, H; (\cdot, U)) && (4)\\
&= \frac{1}{2}\,(10) + \frac{1}{2}\,(0) && (5)\\
&= 5. && (6)
\end{aligned}$$

And, Bob's expected utility would be

$$\begin{aligned}
\mathbb{E}\left[u_b(CH, C)\right] &= \sum_{\mathbf{t} \in T} \Pr(\mathbf{t})\, u_b(CH(t_b), C(t_a); \mathbf{t}) && (7)\\
&= \Pr(I)\, u_a(CH(I), C(\cdot); (\cdot, I)) + \Pr(U)\, u_a(CH(U), C(\cdot); (\cdot, U)) && (8)\\
&= \Pr(I)\, u_b(C, C; (\cdot, I)) + \Pr(U)\, u_b(H, C; (\cdot, U)) && (9)\\
&= \frac{1}{2}\,(5) + \frac{1}{2}\,(10) && (10)\\
&= \frac{15}{2}. && (11)
\end{aligned}$$

We can continue in this fashion to compute all the ex-ante expected utilities in this Bayesian version of Battle of the Sexes, which yields the following normal-form representation of the Bayesian game:

	CC	CH	HC	HH
C	(10, 5/2)	(5, 15/2)	(5, 0)	(0, 5)
H	(0, 5/2)	(5/2, 0)	(5/2, 15/2)	(5, 0)

Figure 3: The expected payoffs in Bayesian Battle of the Sexes.

There are no dominated strategies in this game. Still, this game has one pure-strategy Nash equilibria, (C, CH), which leads to payoffs of 5 for Alice and 15/2 for Bob.

3 *Finding Mixed Strategies*

In addition to the one pure-strategy Nash equilibrium, there, are potentially more equilibria, namely mixed-strategy Nash equilibria.

- Let p be the probability that Alice plays C.

- Let q_I be the probability that Bob plays C, if Bob is interested in Alice.
- Let q_U be the probability that Bob plays C, if Bob is uninterested in Alice.

If Alice uses the mixed strategy p, how should Bob respond? Well, this depends on whether Bob has type I or type U. We will derive two mixed-strategy Nash equilibria, each one corresponding to whether we start by assuming Bob has type I or type U.

3.1 *Starting Point: If Bob is Interested*

If Bob is interested in Alice, his payoff for playing C is:

$$5p + 0(1-p) = 5p. \tag{12}$$

His payoff for playing H is:

$$0p + 10(1-p) = 10 - 10p. \tag{13}$$

What value of p for Alice makes Bob indifferent between his two actions? Equating the payoffs, and solving for p yields:

$$5p = 10 - 10p \tag{14}$$
$$15p = 10 \tag{15}$$
$$p = \frac{2}{3} \tag{16}$$

Therefore, the mixed strategy $(2/3, 1/3)$ for Alice makes Bob indifferent between his two actions, if Bob has type I.

If Bob is uninterested in Alice, his payoff for playing C is:

$$0p + 5(1-p) = 5(1-p) \tag{17}$$

His payoff for playing H is:

$$10p + 0(1-p) = 10p. \tag{18}$$

Plugging in Alice's strategy yields a payoff of $5 - 5\left(\frac{2}{3}\right) = \frac{5}{3}$ for playing C, and $10\left(\frac{2}{3}\right) = \frac{20}{3}$ for playing H. Bob's payoff if he has type U is strictly greater when playing H, so $q_U = 0$.

Alice's payoff for playing C, when $q_U = 0$, is:

$$\Pr(I)\left[10q_I + 0(1-q_I)\right] + \Pr(U)\left[10q_U + 0(1-q_U)\right] \tag{19}$$
$$= \frac{1}{2}\left[10q_I\right] + \frac{1}{2}\left[10q_U\right] \tag{20}$$
$$= \frac{1}{2}\left[10q_I\right] + \frac{1}{2}\left[10(0)\right] \tag{21}$$

$$= 5q_I. \tag{22}$$

Her payoff for playing H, when $q_U = 0$, is:

$$\Pr(I)\,[0q_I + 5(1-q_I)] + \Pr(U)\,[0q_U + 5(1-q_U)] \tag{23}$$
$$= \frac{1}{2}\,[5(1-q_I)] + \frac{1}{2}\,[5(1-q_U)] \tag{24}$$
$$= \frac{1}{2}\,[5(1-q_I)] + \frac{1}{2}\,[5(1)] \tag{25}$$
$$= \frac{5}{2}(1-q_I) + \frac{5}{2}. \tag{26}$$

What value of q_I for Bob makes Alice indifferent between her two actions? Equating the payoffs, and solving for q_I yields:

$$5q_I = \frac{5}{2}(1-q_I) + \frac{5}{2} \tag{27}$$
$$10q_I = 5(1-q_I) + 5 \tag{28}$$
$$15q_I = 10 \tag{29}$$
$$q_I = \frac{2}{3} \tag{30}$$

Putting it all together:

1. Alice plays C with probability $p = \frac{2}{3}$, and H with probability $1 - p = \frac{1}{3}$.
2. If Bob has type I, then he plays C with probability $q_I = \frac{2}{3}$, and H with probability $1 - q_I = \frac{1}{3}$.
3. If Bob has type U, then he plays C with probability $q_U = 0$, and H with probability $1 - q_U = 1$.

We summarize this mixed strategy as follows:

$$\left(\underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{\text{Alice}}, \underbrace{\left(\underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{\text{Type I}}, \underbrace{(0,1)}_{\text{Type U}} \right)}_{\text{Bob}} \right). \tag{31}$$

So Bob plays strategy CH with probability 2/3, and strategy HH with probability 1/3. Mixing in Alice's strategy yields the following joint distribution over Alice's and Bob's strategies at this equilibrium:

	CC	CH	HC	HH
C	0	4/9	0	2/9
H	0	2/9	0	1/9

Figure 4: The joint probabilities at this mixed-strategy Nash equilibirum.

At this equilibrium, Alice's utility is 30/9 and Bob's is 35/9.

Verification We now verify that this mixed strategy is in fact a Bayes-Nash equilibrium. Fixing Alice's (Bob's) strategy, it should be the case that Bob (Alice) cannot employ an alternative mixed strategy that yields strictly more utility.

Alice The expected utility Alice receives for playing C is:

$$u_a(C) = \Pr(I)\left[10\left(q_I\right) + 0\left(1 - q_I\right)\right] + \Pr(U)\left[10\left(q_U\right) + 0\left(1 - q_U\right)\right] \tag{32}$$

$$= \frac{1}{2}\left[10\left(\frac{2}{3}\right) + 0\left(\frac{1}{3}\right)\right] + \frac{1}{2}\left[10\left(0\right) + 0\left(1 - 0\right)\right] \tag{33}$$

$$= \frac{10}{3}. \tag{34}$$

The expected utility Alice receives for playing H is:

$$u_a(H) = \Pr(I)\left[0\left(q_I\right) + 5\left(1 - q_I\right)\right] + \Pr(U)\left[0\left(q_U\right) + 5\left(1 - q_U\right)\right] \tag{35}$$

$$= \frac{1}{2}\left[0\left(\frac{2}{3}\right) + 5\left(\frac{1}{3}\right)\right] + \frac{1}{2}\left[0\left(0\right) + 5\left(1 - 0\right)\right] \tag{36}$$

$$= \frac{10}{3}. \tag{37}$$

Since the expected utilities are equal, Alice is indifferent between playing C and H, and cannot improve her expected utility by mixing.

Bob (type I) The expected utility Bob (type I) receives for playing C is

$$u_b(C) = 5\left(p\right) + 0\left(1 - p\right) \tag{38}$$

$$= 5p \tag{39}$$

$$= \frac{10}{3}. \tag{40}$$

The expected utility Bob (type I) receives for playing H is

$$u_b(H) = 0\left(p\right) + 10\left(1 - p\right) \tag{41}$$

$$= 10\left(1 - p\right) \tag{42}$$

$$= \frac{10}{3}. \tag{43}$$

Since the expected utilities are equal, Bob is indifferent between playing C and H, and cannot improve his expected utility by mixing.

Bob (type U) The expected utility Bob (type U) receives for playing C is

$$u_b(C) = 0\left(p\right) + 5\left(1 - p\right) \tag{44}$$

$$= 5(1-p) \tag{45}$$
$$= \frac{5}{3}. \tag{46}$$

The expected utility Bob (type U) receives for playing H is

$$u_b(H) = 10\,(p) + 0\,(1-p) \tag{47}$$
$$= 10\,(p) \tag{48}$$
$$= \frac{20}{3}. \tag{49}$$

Since the expected utility of playing H is strictly larger than the expected utility of playing C, Bob will play H.

3.2 Starting Point: If Bob is Uninterested

If Bob is uninterested in Alice, his payoff for playing C is:

$$0p + 5(1-p) = 5(1-p). \tag{50}$$

His payoff for playing H is:

$$10p + 0(1-p) = 10p. \tag{51}$$

What value of p for Alice makes Bob indifferent between his two actions? Equating the payoffs, and solving for p yields:

$$5 - 5p = 10p \tag{52}$$
$$5 = 15p \tag{53}$$
$$p = \frac{1}{3} \tag{54}$$

Therefore, the mixed strategy $(1/3, 2/3)$ for Alice makes Bob indifferent between his two actions, if Bob has type U.

If Bob is interested in Alice, his payoff for playing C is:

$$5p + 0(1-p) = 5p. \tag{55}$$

His payoff for playing H is

$$0p + 10(1-p) = 10(1-p). \tag{56}$$

Plugging in Alice's strategy yields a payoff of $5\left(\frac{1}{3}\right) = \frac{5}{3}$ for playing C, and $10\left(\frac{2}{3}\right) = \frac{20}{3}$ for playing H. Bob's payoff if he has type I is strictly greater when playing H, so $q_I = 0$.

Alice's payoff Alice for playing C, when $q_I = 0$, is:

$$\Pr(I)\left[10q_I + 0(1-q_I)\right] + \Pr(U)\left[10q_U + 0(1-q_U)\right] \tag{57}$$
$$= \frac{1}{2}\left[10q_I\right] + \frac{1}{2}\left[10q_U\right] \tag{58}$$

$$= \frac{1}{2}\left[10(0)\right] + \frac{1}{2}\left[10q_U\right] \tag{59}$$
$$= 5q_U. \tag{60}$$

Her payoff for playing H, when $q_I = 0$, is:

$$\Pr(I)\left[0q_I + 5(1-q_I)\right] + \Pr(U)\left[0q_U + 5(1-q_U)\right] \tag{61}$$
$$= \frac{1}{2}\left[5(1-q_I)\right] + \frac{1}{2}\left[5(1-q_U)\right] \tag{62}$$
$$= \frac{1}{2}\left[5(1)\right] + \frac{1}{2}\left[5(1-q_U)\right] \tag{63}$$
$$= \frac{5}{2} + \frac{5}{2}(1-q_U). \tag{64}$$

Equating the payoffs, we solve for q_U:

$$5q_U = \frac{5}{2} + \frac{5}{2}(1-q_U) \tag{65}$$
$$10q_U = 5 + 5(1-q_U) \tag{66}$$
$$15q_U = 10 \tag{67}$$
$$q_U = \frac{2}{3} \tag{68}$$

Putting it all together:

1. Alice plays C with probability $p = \frac{1}{3}$, and H with probability $1 - p = \frac{2}{3}$.
2. If Bob has type I, then he plays C with probability $q_I = 0$, and H with probability $1 - q_I = 1$.
3. If Bob has type U, then he plays C with probability $q_U = \frac{2}{3}$, and H with probability $1 - q_U = \frac{1}{3}$.

We summarize this mixed strategy as follows:

$$\left(\underbrace{\left(\frac{1}{3}, \frac{2}{3}\right)}_{\text{Alice}}, \underbrace{\left(\underbrace{(0,1)}_{\text{Type I}} \underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{\text{Type U}}, \right)}_{\text{Bob}} \right). \tag{69}$$

So Bob plays strategy HC with probability $2/3$, and strategy HH with probability $1/3$. Mixing in Alice's strategy yields the following joint distribution over Alice's and Bob's strategies at this equilibrium:

At this equilibrium, Alice's utility is $30/9$ and Bob's is $35/9$.

	CC	CH	HC	HH
C	0	0	2/9	1/9
H	0	0	4/9	2/9

Figure 5: The joint probabilities at this mixed-strategy Nash equilibirum.

Verification We now verify that this mixed strategy is in fact a Bayes-Nash equilibrium. Fixing Alice's (Bob's) strategy, it should be the case that Bob (Alice) cannot employ an alternative mixed strategy that yields strictly more utility.

Alice The expected utility Alice receives for playing C is

$$u_a(C) = \Pr(I)\left[10(q_I) + 0(1-q_I)\right] + \Pr(U)\left[10(q_U) + 0(1-q_U)\right] \tag{70}$$

$$= \frac{1}{2}\left[10(0) + 0(1)\right] + \frac{1}{2}\left[10\left(\frac{2}{3}\right) + 0\left(\frac{1}{3}\right)\right] \tag{71}$$

$$= \frac{10}{3}. \tag{72}$$

The expected utility Alice receives for playing H is

$$u_a(H) = \Pr(I)\left[0(q_I) + 5(1-q_I)\right] + \Pr(U)\left[0(q_U) + 5(1-q_U)\right] \tag{73}$$

$$= \frac{1}{2}\left[0(0) + 5(1)\right] + \frac{1}{2}\left[0\left(\frac{2}{3}\right) + 5\left(\frac{1}{3}\right)\right] \tag{74}$$

$$= \frac{10}{3}. \tag{75}$$

Since the expected utilities are equal, Alice is indifferent between playing C and H, and cannot improve her expected utility by mixing.

Bob (type I) The expected utility Bob (type I) receives for playing C is

$$u_b(C) = 5(p) + 0(1-p) \tag{76}$$

$$= 5p \tag{77}$$

$$= \frac{5}{3}. \tag{78}$$

The expected utility Bob (type I) receives for playing H is

$$u_b(H) = 0(p) + 10(1-p) \tag{79}$$

$$= 10(1-p) \tag{80}$$

$$= \frac{20}{3}. \tag{81}$$

Since the expected utility of playing H is strictly larger than the expected utility of playing C, Bob will play H.

Bob (type U) The expected utility Bob (type U) receives for playing C is

$$
\begin{aligned}
u_b(C) &= 0\,(p) + 5\,(1-p) && (82)\\
&= 5(1-p) && (83)\\
&= \frac{10}{3}. && (84)
\end{aligned}
$$

The expected utility Bob (type U) receives for playing H is

$$
\begin{aligned}
u_b(H) &= 10\,(p) + 0\,(1-p) && (85)\\
&= 10\,(p) && (86)\\
&= \frac{10}{3}. && (87)
\end{aligned}
$$

Since the expected utilities are equal, Bob is indifferent between playing C and H, and cannot improve his expected utility by mixing.