CS 1951-k and CS 2951-z: Homework 9

Professor Greenwald

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Due Date: Tuesday, April 28, 2020. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course (CS1951k/CS2951z) you are taking.

For 1000-level, you should solve the first three problems. For 2000-level credit, you must solve all four.

Unit-demand Bidders

Throughout this assignment, we assume the following multi-parameter setting:

1. There is a set G of m (possibly heterogeneous) goods.
2. There is a set N of n bidders, with each bidder i characterized by a private valuation v_i that ascribes value v_{ij} to good j.
3. The bidders' valuations are characterized as **unit-demand**, meaning each bidder i values each bundle $X \subseteq G$ as $v_i(X) = \max_{j \in X} v_{ij}$.

An outcome in this model consists of an allocation and payment scheme (also called a **pricing**). An allocation is a matching M of goods to bidders, in which each bidder is matched to at most one good, and each good, to at most one bidder. A pricing is a vector of prices $q \in \mathbb{R}_+^m$, where $q(j)$ is the price of good $j \in G$. We denote the good matched to bidder i under matching M by $M(i) \in G$, and its price by $q(M(i)) \in \mathbb{R}_+$. A bidder i might not be matched in M, in which case we define $M(i) = \emptyset$ and $v_i(\emptyset) = q(\emptyset) = 0$. The welfare of a matching M is defined as $W(M) = \sum_i v_i(M(i))$. Finally, we define a **Walrasian equilibrium** (WE) (M, q) as follows:

WE1 Each bidder i is allocated a preferred good: i.e., one such that

$$v_i(M(i)) - q(M(i)) \geq v_i(j) - q(j), \quad \forall j \in G.$$

Note that this condition implies that all bidders' utilities are non-negative, since $M(i) = \emptyset$ is a valid allocation for bidder i.

WE2 The market clears: i.e., if good j is unallocated, then $q(j) = 0$. Likewise, if $q(j) > 0$, then there exists an i such that $M(i) = j$.

1 *Walras' Law*

Given a unit-demand market together with a matching M and a pricing q, the the total value of demand is $\sum_{i \in N} q(M(i))$ and the total value of supply is $\sum_{j \in G} q(j)$ Walras' Law[1] asserts that, at (Walrasian) equilibrium:

$$\sum_{i \in N} q(M(i)) = \sum_{j \in G} q(j)$$

Prove that Walras' Law is equivalent to the market clearance condition (WE2).

[1] Léon Walras was a French economist who pioneered the development of general equilibrium theory.

2 *Mixing and Matching Matchings*

Prove the following claim: if (M, q) is a Walrasian equilibrium and $M^* \neq M$ is a welfare-maximizing matching, then (M^*, q) is a Walrasian equilibrium. **Hint**: Use the First Welfare Theorem.

3 *Walrasian Equilibrium Prices form a Lattice*

Prove that Walrasian Equilibrium price vectors form a lattice. Concretely, let (M, q^1) and (M, q^2) be two Walrasian equilibria. Show that $(M, q^1 \wedge q^2)$ and $(M, q^1 \vee q^2)$ are also Walrasian equilibria, where $q^1 \wedge q^2$ and $q^1 \vee q^2$ are pricings obtained by taking the component-wise minimum and maximum of q^1 and q^2, respectively: i.e.,

- $(q^1 \wedge q^2)(j) = \min\{q_1(j), q_2(j)\}$
- $(q^1 \vee q^2)(j) = \max\{q_1(j), q_2(j)\}$

4 *ϵ-Walrasian Equilibrium and the First Welfare Theorem*

Define an ϵ-Walrasian Equilibrium (M, q) as follow:

WE1 Each bidder i is allocated a preferred good: i.e., one such that

$$v_i(M(i)) - q(M(i)) \geq v_i(j) - q(j) - \epsilon, \quad \forall j \in G,$$

and all bidders' utilities are non-negative.

WE2 The market clears: i.e., if good j is unallocated, then $q(j) = 0$. Likewise, if $q(j) > 0$, then there exists an i such that $M(i) = j$.

Prove the following approximate version of the celebrated First Welfare Theorem: if (M, q) is an ϵ-Walrasian equilibrium, then $W(M)$ is within $\epsilon \min\{n, m\}$ of the value of a welfare-maximizing matching.