Homework 07: Simple, Approximate Mechanisms

CS 1951k/2951z

2020-04-09

Due Date: Tuesday, April 21, 2020. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course (CS1951k/CS2951z) you are taking.

For 1000-level credit, you need only solve the first two problems, and the first two parts of the third problem (but you should be sure to read the third part). For 2000-level credit, you should solve all three problems.

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1 *The Vickrey Auction: An Oldie but Goodie*

Consider a single-good auction, and assume bidders' values are drawn *i.i.d.* from a regular distribution F. Prove that the expected revenue of the Vickrey auction (second-price without a reserve) with n bidders, V_n, is at least $^{n-1}/_n$ times that of Myerson's optimal auction (second-price with the monopoly price as the reserve) with the same number of bidders. **Hint:** Use the Bulow-Klemperer Theorem.

2 *The Prophet Inequality is Tight*

Prove that the 2-approximation given by the prophet inequality is tight. That is, establish a corresponding upper bound by showing that there exist independent distributions $G_1, \ldots, G_n$ for $n \geq 2$ for which there does not exist a strategy (threshold or otherwise) that accrues higher reward than $1/2$ that of the prophet.

Hint: Exhibit independent distributions $G_1, \ldots, G_n$ such that for all $\epsilon > 0$, all strategies (threshold or otherwise) yield expected value strictly less than $(1/2 + \epsilon)\, E_{\boldsymbol{\pi} \sim \mathbf{G}} \left[\max_i \pi_i\right]$, where $E_{\boldsymbol{\pi} \sim \mathbf{G}} [\max_i \pi_i]$ is the expected reward of the prophet.

3 *A Dynamic Threshold Strategy*

Recall the game that motivated the Prophet Inequality: Assume n independent random variables π with non-negative, continuous distributions G_i. The distributions G_i are known in advance, but the "prize" π_i is not revealed until period i. The player can then choose to accept π_i and end the game, or reject it and continue playing.

In lecture, we proved that the following (static) thresholding strategy is guaranteed to obtain at least half the expected value of an omniscient prophet who knows all the prizes π_i in advance:

1. Select a threshold value t s.t. $p(t) = \Pr(\pi_i < t \, \forall 1 \leq i \leq n) = 1/2$.
2. Accept the first prize π_i, if any, that exceeds t.

In this problem, we will consider a dynamic variant of this strategy, and analyze its performance. The dynamic variant is as follows:

1. If one rejects the prize π_i at period i, then compute a *new* threshold t_i s.t. $\Pr(\pi_j < t_i \, \forall j > i) = 1/2$.
2. Accept any prize π_i that is greater than the current threshold t_i.

In answering the first two questions, you can assume an i.i.d. setting, with $G_i = \text{Uniform}(0, 1)$, for all $i \in \{1, \ldots, n\}$.

1. Compute t as per the static thresholding strategy, as a function of n. Additionally, compute the expected value of the static thresholding strategy.
2. Compute the expected value of the dynamic thresholding strategy, as a function of n. How does this expected value compare with that of the static strategy?

   Hint: For this question, you may plot the expected values of the static and dynamic strategies for various values of n instead of computing the expected value analytically.
3. Is the dynamic strategy guaranteed to weakly outperform the static one in the i.i.d. setting (not necessarily Uniform(0, 1)? Provide either a proof or a counterexample.