Homework 6: Posted-Price Mechanisms

CS 1951k/2951z

2020-03-05

Due Date: Tuesday, March 10, 2020. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course (CS1951k/CS2951z) you are taking.

For 1000-level credit, you need only solve the first four problems. For 2000-level credit, you must solve all five.

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1 *An Equal-Revenue Distribution*

Suppose all bidders draw their values for a single good from the following distribution: $F_i(v_i) = 1 - 1/v_i$, for all $i \in N$, $v_i \in T_i = [1, \infty)$.

1. What virtual value function corresponds to this distribution?
2. How would Myerson's optimal auction allocate this good? What would each bidder pay to satisfy IC and IR?
3. What revenue curve $R(\pi)$ corresponds to this distribution, when the posted price $\pi > 1$. **Hint:** Distributions of this form are sometimes called **equal-revenue distributions**.
4. Why isn't Myerson's auction optimal in this setting? Describe an IC and IR auction that achieves greater expected revenue than Myerson's auction given this distribution.

2 *Revenue Curves and the Median*

The revenue curve for a distribution F as a function of q is given by:

$$R(q) = q\, F^{-1}\,(1 - q)\,.$$

Assume that F is regular and has strictly positive density over its support. Let $q(v)$ be the quantile of value v; likewise, let $v(q)$ be the value of quantile q. For example, if the median of distribution F is κ, then $F(\kappa) = 1/2$, $F^{-1}(1/2) = \kappa$, $v(1/2) = \kappa$, and $q(\kappa) = 1/2$.

1. Prove that the value of the revenue curve at quantile $q = 1/2$ is $\kappa/2$: i.e., $R(1/2) = \kappa/2$, where κ is the median.
2. Prove that the revenue curve is upper-bounded by κ: i.e., $R(q) \leq \kappa$, for all $q \in [0, 1]$.

3 *Posted-Price vs. Second-Price Revenue*

Prove that the ratio of the expected revenue of the posted-price mechanism for a single good and n bidders, with posted price $\pi = F^{-1}\left(1 - 1/n\right)$, to that of the second-price auction is:

$$\frac{\text{PP}}{\text{2nd}} \geq 1 - \frac{1}{e}$$

Assume the bidders' values are drawn from the uniform distribution on $[0, 1]$. **Hint:** $(1 - 1/n)^n \leq 1/e$, for all $n \geq 1$.

4 *Optimizing Posted Prices*

This problem is designed to give you some intuition about why analyzing the expected welfare of a posted-price mechanism is harder than analyzing its expected revenue.

Assume n bidders are participating in a posted-price mechanism for a single indivisible good. Assume further that the bidders' values for the good are drawn from the uniform distribution on $[0, 1]$,

1. Suppose the goal is to maximize expected revenue. Derive an expression for the optimal posted price $\pi^* \in [0, 1]$ in closed form.
2. Suppose the goal is to to maximize expected welfare. Argue that the optimal posted price is 0 when $n = 1$. What is the optimal posted price when $n = 2$? And what do think happens for $n \geq 3$?
3. Give an example of a value distribution F over valuations s.t. the welfare-maximizing posted price definitively *not* 0, for all $n > 1$, assuming the n bidders values' are drawn i.i.d. from F.

5 *Approximately Optimal Second-Price Auctions*

In this problem, you will analyze the **single sample auction**, a second-price auction with a sample reserve price. It works as follows:

> Bids are collected. A reserve price is chosen by removing an arbitrary bidder j from the auction, and setting the reserve price to be j's bid. The auctioneer then allocates the good to the bidder with highest bid iff their bid is at least this reserve, and charges the winner, if any, the greater of the second-highest bid and the reserve price.

Your goal is to show that this auction achieves $1/2\left(n-1/n\right)$ of the optimal revenue, where n is the number of bidders, assuming symmetic values drawn from a regular distribution F.

1. Let t and v be two values in the support of regular distribution F. Let r^* be the monopoly reserve price. Show that

$$\mathop{\mathbb{E}}_{v \sim F} \left[R \left(\max\{t, v\} \right) \right] \geq 1/2 \, R \left(\max\{t, r^*\} \right).$$

2. Let APX denote the expected revenue generated by the single sample auction. Let OPT denote the expected revenue generated by the optimal (i.e., revenue-maximizing) auction. Show that the single sample auction generates, in expectation, approximately half the total expected revenue generated by the optimal auction: i.e.,

$$\frac{\text{APX}}{\text{OPT}} \geq \frac{1}{2} \left(\frac{n-1}{n} \right).$$