Homework 5: Myerson's Theorem

CS 1951k/2951z

2020-02-27

Due Date: Tuesday, March 3, 2020. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course (CS1951k/CS2951z) you are taking.

For 1000-level credit, you need only solve the first three problems. For 2000-level credit, you must solve the first four problems.

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1 *Bayesian Constraints, continued*

Recall the Bayesian (i.e., interim) formulation of the auction design problem from Homework 4. Since the interim constraints are weaker than the dominant-strategy constraints presented in lecture, you might imagine that the welfare achieved by the welfare-maximizing auction in the interim case exceeds that of the dominant strategy case. Argue that this is not in fact the case: i.e., that the value that maximizes expected welfare in this model is the same regardless of whether the IC and IR constraints are interim or ex-post. (The same holds for revenue, but you need only make the argument once.)

2 *Revenue Equivalence*

In this problem, you will explore the expected revenue of different auction formats. Answer the questions, assuming two bidders whose valuations are both drawn from a uniform $U[0,1]$ distribution.

1. What is the expected revenue of Myerson's optimal (i.e., revenue-maximizing) auction?

   Hints: What is each bidder i's virtual value function φ_i? Is this virtual value function non-decreasing (i.e., is $U[0,1]$ regular)? What is the inverse of this virtual value function, and what is the reserve price?

2. Derive the expected revenue of a second-price auction with this same reserve.

3. Use Myerson's payment formula to explain this equivalence.

3 *Sponsored Search: Revenue*

As in Homework 3, assume n bidders (online advertisers) are competing for one of k slots on a page that results from a keyword search (e.g., "TV"). Each slot can be allocated to at most one bidder, and each bidder can be allocated at most one slot.

For each slot j, there is an associated probability that a user conducting an organic search will click on an ad in that slot. This probability is called the **click-through-rate** (CTR).[1] For slot j, we denote the CTR by α_j, and we assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$.

[1] In reality, the probability a user clicks on an ad depends on both its position *and* its relevance.

Each bidder i also has a private value v_i that corresponds to how much they value a user clicking on their ad (e.g., an estimate of how much they expect to profit per click). Thus, if an bidder is allocated slot j (i.e., $x_i = \alpha_j$) and pays p_i, their utility is given by $u_i = \alpha_j v_i - p_i$.

Design a sponsored search auction (i.e., an allocation scheme and a payment rule) for slots on a web page that collects one bid from each bidder, and then allocates each slot to at most one bidder, and at most one slot to each bidder. Your auction should maximize revenue and satisfy the usual constraints of individual rationality, incentive compatibility, and ex-post feasibility. Use Myerson's lemma to argue and theorem that your auction satisfies these requirements.

4 *Another Auction with a Reserve*

Reserve prices are necessary to maximize expected revenue, since they given auctioneers the flexibility to charge more money to bidders whom they expect to have high values, while still preserving incentive compatibility and individual rationality.

Bob understands this, but he does not understand why the revenue-maximizing auction for a single good would allocate to the bidder with the highest virtual value, rather than the bidder with the highest value. To him, it seems like the revenue-maximizing auction should allocate to the bidder with the highest value, since that bidder is willing to pay the most! So Bob proposes the following revision to Myerson's revenue-maximizing single-good auction: use the same reserve prices as before, but allocate to the bidder with the highest value among the bidders who bid above their reserve prices.

To be clear, Bob's allocation rule works as follows:

- Assume $v_i \sim F_i$ for all bidders $i \in N$, where F_i is regular with bounded support.
- Determine the set C of bidders i who bid at least their reserve, namely $\phi_i^{-1}(0)$, where ϕ_i is bidder i's virtual value function.
- Allocate the good to the highest bidder in C.

1. Derive the payment rule that creates an auction that satisfies IC and IR with Bob's allocation rule. Prove that Bob's allocation rule together with this payment rule satisfies IC and IR.

2. When 0 or 1 bidders bid above their reserve prices, Bob's and Myerson's auctions yield the exact same outcome.

   Consider the case in which there are at least two bidders who bid above their reserve prices, and among them, the bidder with the highest value is not the same as the bidder with the highest virtual value. Prove that Bob's auction yields weakly greater expected revenue than Myerson's auction in this case.

3. Consider the other case, in which the bidder with the highest value also has the highest virtual value. In this case:

   (a) Provide an example in which Bob's auction produces greater revenue.

   (b) Provide an example in which Myerson's auction produces greater revenue.

   Your examples should define the number of bidders, each bidder's valuation distribution, and each bidder's value.

4. Prove that if all bidders have the same valuation distribution, and their (shared) virtual value function is strictly increasing, Bob's and Myerson's auctions are exactly the same: i.e., the same bidder wins, and they pay the same price.

5. Suppose that there are just two bidders. Bidder A whose value is drawn from $U(0,1)$ and Bidder B whose value is drawn from $U(0,2)$. Compute the expected revenue of both auctions.

   Hint: Setup cases based on the bidders' values and virtual values.