Homework 4: Myerson's Lemma

CS 1951k/2951z

2020-02-20

Due Date: Tuesday, February 25, 2020. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course (CS1951k/CS2951z) you are taking.

For 1000-level credit, you need only solve the first three problems. For 2000-level credit, you must solve all four problems.

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1 *Bayesian Constraints*

Rather than insisting that incentive compatibility and individual rationality hold *always*, suppose we relax these requirements and ask only that these properties hold *in expectation*.

Define the **interim allocation** and **interim payment** functions, respectively, as follows:

$$\hat{x}_i(v_i) = \mathop{\mathbb{E}}_{\mathbf{v}_{-i} \sim F_{-i}} \left[x_i(v_i, \mathbf{v}_{-i}) \right], \quad \forall i \in N, \forall v_i \in T_i, \tag{1}$$

$$\hat{p}_i(v_i) = \mathop{\mathbb{E}}_{\mathbf{v}_{-i} \sim F_{-i}} \left[p_i(v_i, \mathbf{v}_{-i}) \right], \quad \forall i \in N, \forall v_i \in T_i. \tag{2}$$

Further, define **Bayesian incentive compatibility** (BIC) to mean that bidding truthfully is, in expectation, utility maximizing:

$$v_i \hat{x}_i(v_i) - \hat{p}_i(v_i) \geq v_i \hat{x}_i(t_i) - \hat{p}_i(t_i), \quad \forall i \in N, \forall v_i, t_i \in T_i. \tag{3}$$

Likewise, define **interim individual rationaltiy** (IIR) to mean that bidding truthfully, in expectation, leads to non-negative utility:

$$v_i \hat{x}_i(v_i) - \hat{p}_i(v_i) \geq 0, \quad \forall i \in N, \forall v_i, t_i \in T_i. \tag{4}$$

Myerson's lemma generalizes to the interim case, so a mechanism satisfies BIC and IIR iff

1. Interim allocations are monotone non-decreasing:

$$\hat{x}_i(v_i) \geq \hat{x}_i(t_i), \quad \forall i \in N, \forall v_i \geq t_i \in T. \tag{5}$$

2. Payments take the following form:

$$\hat{p}_i(v_i) = v_i \hat{x}_i(v_i) - \int_0^{v_i} \hat{x}_i(z) \, \mathrm{d}z, \quad \forall i \in N, \forall v_i \geq t_i \in T. \tag{6}$$

Suppose that we are designing a mechanism to auction off one item in a sealed-bid format, just like in first- and second-price auctions. The value v_i of bidder $i \in N$, is a sample from distribution F_i, i.e., $v_i \sim F_i, \forall i \in N$, with all n values drawn independently.

1. The interim allocation at v_i is the probability of winning with value v_i (assuming truthful bidding on the part of the others). Simplify the interim allocation function, and explain its meaning in words.
2. Calculate the interim allocation function $\hat{x}_i(v_i)$, if $n = 2$ bidders and each bidder draws their values from a $U(0,1)$ distribution. Show your work.
3. What is the (closed-form) payment formula when there are two bidders, each drawing value from a $U(0,1)$ distribution? Show your work.
4. **Extra Credit:** Given n bidders and $U(0,1)$ distributions, give the analytic form of the interim allocation and payment functions. Show your work.

2 *Allocation Rule Discontinuity*

Fix a bidder i and a profile $\mathbf{v}_{-i}$. Myerson's lemma tells us that incentive compatibility and individual rationality imply two properties:

1. Allocation monotonicity: one's allocation should not decrease as one's value v_i increases.
2. Myerson's payment formula:

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z, \quad \forall i \in N, \forall v_i \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \tag{7}$$

In a second-price auction, the allocation rule is piecewise constant on any continuous interval. That is, bidder i's allocation function is a **Heaviside step function**,[1] with discontinuity at $v_i = b^*$, where b^* is the highest bid among all bidders other than i (i.e., $b^* = \max_{j \in N \setminus i} v_j$):

[1] This is the canonical step function, whose range is $[0,1]$.

$$x_i(v_i, \mathbf{v}_{-i}) = \begin{cases} 1, & \text{if } v_i > b^* \\ 1/2, & \text{if } v_i = b^* \\ 0, & \text{otherwise.} \end{cases}$$

(In writing $1/2$, we assume all bids other than i's are unique.)

Given this allocation rule, the payment formula tells us what i should pay, should they be fortunate enough to win:

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \\ &= v_i(1) - \left(\int_0^{b^*} 0\, \mathrm{d}z + \int_{b^*}^{v_i} 1\, \mathrm{d}z \right) \\ &= v_i(1) - (0 + v_i - b^*) \\ &= b^*. \end{aligned}$$

Alternatively, i's payment can be expressed as follows: for any arbitrarily small $\epsilon > 0$,

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= \int_0^{v_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) \mathrm{d}z \\ &= \int_0^{b^*} z \left(\frac{\mathrm{dx_i(z, \mathbf{v}_{-i})}}{\mathrm{d}z} \right) \mathrm{d}z + \int_{b^*}^{b^*} z \left(\frac{\mathrm{dx_i(z, \mathbf{v}_{-i})}}{\mathrm{d}z} \right) \mathrm{d}z + \int_{b^*}^{1} z \left(\frac{\mathrm{dx_i(z, \mathbf{v}_{-i})}}{\mathrm{d}z} \right) \mathrm{d}z \\ &= \int_0^{b^*-\epsilon} z(0)\, \mathrm{d}z + \int_{b^*-\epsilon}^{b^*+\epsilon} z \left(\frac{\mathrm{dx_i(z, \mathbf{v}_{-i})}}{\mathrm{d}z} \right) \mathrm{d}z + \int_{b^*+\epsilon}^{1} z(0)\, \mathrm{d}z \\ &= \int_{b^*-\epsilon}^{b^*+\epsilon} z \left(\frac{\mathrm{dx_i(z, \mathbf{v}_{-i})}}{\mathrm{d}z} \right) \mathrm{d}z \\ &= b^* \cdot [\text{jump in } x_i(\cdot, \mathbf{v}_{-i}) \text{ at } b^*]. \end{aligned}$$

Suppose that the allocation rule is piecewise constant on the continuous interval $[0, v_i]$, and discontinuous at points $\{z_1, z_2, \ldots, z_\ell\}$ in this interval. That is, there are ℓ points at which the allocation jumps from $x(z_j, \mathbf{v}_{-i})$ to $x(z_{j+1}, \mathbf{v}_{-i})$ (see Figure 1). Assuming this "jumpy" allocation rule is monotone non-decreasing in value, prove that Myerson's payment rule can be expressed as follows:

$$p_i(v_i, \mathbf{v}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \left[\text{jump in } x_i(\cdot, \mathbf{v}_{-i}) \text{ at } z_j \right]. \tag{8}$$

3 *Sponsored Search Extension*

In this problem, we generalize our model of sponsored search to include an additional *quality* parameter $\beta_i > 0$ that characterizes each bidder i. With this additional parameter, we can view α_j as the probability a user views an ad, and β_i as the conditional probability that a user then clicks, given that they are already viewing the ad. Note that α_j, the view probability, depends only on the slot j, not on the advertiser occupying that slot, while β_i, the conditional click probablity, explicitly depends on the advertiser i.

In this model, given bids $\mathbf{v}$, bidder i's utility is given by:

$$u_i(\mathbf{v}) = \beta_i v_i x(\mathbf{v}) - p(\mathbf{v})$$

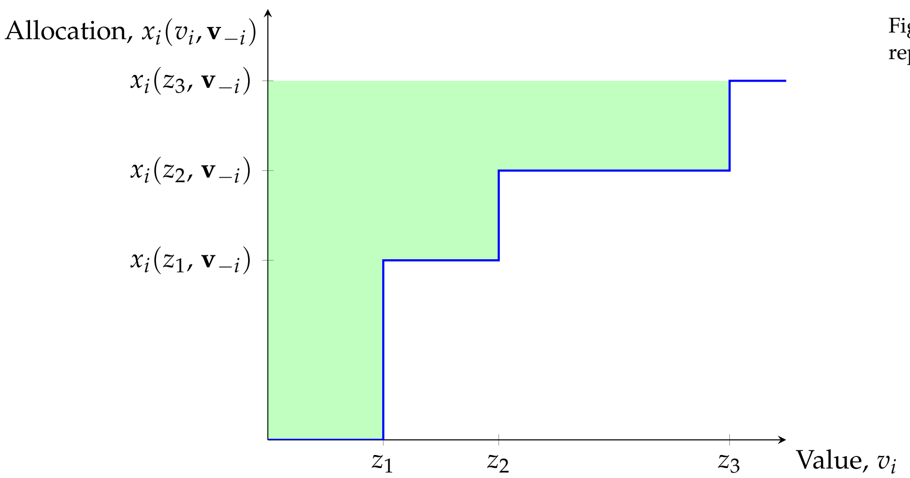

Figure 1: Allocation Rule. Shaded area represents payment.

So if bidder i is allocated slot j, their utility is:

$$u_i(\mathbf{v}) = \beta_i v_i \alpha_j - p(\mathbf{v})$$

You should assume qualities are public, not private, information.

1. Define welfare in this setting, and then describe an allocation rule that maximizes the welfare. Justify your answer.

2. Argue that your allocation rule is monotonic, and then use Myerson's formula to produce a payment rule that yields a DSIC mechanism.

4 *The Knapsack Auction*

The *knapsack problem* is a famous NP-hard[2] problem in combinatorial optimization. The problem is stated as follows:

> There is a knapsack, which can hold a maximum weight of W. There are n items; each item i has weight $w_i \leq W$ and value v_i. The goal is to find some subset S of items of maximal total value with total weight no more than W. Written as an integer program,
>
> $$\max \sum_{i=1}^{n} x_i v_i$$
>
> subject to
>
> $$\sum_{i=1}^{n} x_i w_i \leq W, x \in \{0, 1\}.$$

[2] There are no known polynomial-time solutions.

We can frame this problem as a mechanism design problem as follows. Each bidder has an item that they would like to put in the knapsack. Each item has a public parameter w_i and a private value

v_i. An auction will take place, after which some set S of bidders will place their items (of total weight less than W) in the knapsack and pay some amount of money to the auctioneer. One possible application of this is auctioning off commercial snippets in a 5-minute ad break[3].

[3] Here, the weight is the time of the commerical in seconds.

Since the problem is NP-hard, we cannot hope to find a polynomial-time welfare-maximizing solution. Instead, we will produce a polynomial-time, DSIC mechanism that is a 2-approximation[4] of the optimal welfare.

[4] Meaning that for any set possible set of valuations and weights, we always achieve at least 50% of the optimal welfare.

We propose the following greedy allocation scheme: Sort the bidders' items by their ratios $\frac{v_i}{w_i}$ and allocate items in that order until there is no room left in the knapsack.

1. Show that this allocation scheme is not a 2-approximation by producing an example where it fails to achieve 50% of the optimal welfare.

2. Alice proposes a small improvement to the greedy allocation scheme. Her improved allocation scheme compares 1) the welfare achieved by the greedy allocation scheme to 2) the welfare achieved by simply putting the single item of highest value into the knapsack and stopping.[5] Her allocation scheme then uses whichever of the two approaches maximizes welfare.

   [5] Note that a weakly greater welfare could be achieved by greedily filling the knapsack with items in decreasing order of value. We do not analyze this because it is more complicated and unnecessary to achieve a 2-approximation.

   It can be proven (indeed, you will prove later in the course) that Alice's combined allocation scheme is now a 2-approximation of optimal welfare.

   Now, we would like to use this allocation scheme to create a mechanism that satisfies individual rationality and incentive compatibility. To do this, we will use Myerson's theorem.

   First, argue that Alice's allocation scheme is monotone.

3. Use Myerson's payment formula to produce payments such that the resulting mechanism is DSIC.