Homework 2: Introduction to Auctions

CS 1951k/2951z

2020-01-30

Due Date: Tuesday, February 4, 2020. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course (CS1951k/CS2951z) you are taking.

For 1000-level credit, you need only solve the first four problems. For 2000-level credit, you must solve all five problems. The *All-Pay Auction* problem is just for fun.

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1 *Bayesian Prisoners' Dilemma*

Mr. Burns: Remember, your job depends on your successful completion of Nuclear Physics 101. Oh, and one more thing...
Mr. Burns: You must find the Jade Monkey before the next full moon.
Smithers: Actually, sir, we found the Jade Monkey. It was in your glove compartment.
Mr. Burns: And the road maps, and ice scraper?
Smithers: They were in there, too, sir.
Mr. Burns: Excellent! It's all falling into place...
-The Simpsons, Season 5, Episode 3: Homer Goes to College

Alice and Bob devise a plan to steal the jade monkey before the next full moon. They realize that it's in a glove compartment, and decide to use a 3d-printed key to break into it. Alice, unfortunately, forgets to bring the key, so the pair gets caught and put into jail.

Alice and Bob are jailed in different cells, and cannot communicate with one another. The prosecutor tells each of the two that they can either: 1. stay silent, or 2. rat out the other.

We say that Alice and Bob cooperate (C) if they choose to stay silent. We say that Alice and Bob defect (D) if they choose to betray the other. The setup so far is identical to that of the standard Prisoners' Dilemma, whose payoff matrix is shown in Figure 1, where Alice is the row player, and Bob is the column player.

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Figure 1: The payoff matrix for the Prisoners' Dilemma.

But, there's a new wrinkle. Alice and Bob are not just thieves; they are also students of moral philosophy. Alice, in particular, is *fair* and makes decisions using John Rawls's maximin criterion. That is, when she makes decisions whose outcomes affect multiple people, she evaluates each outcome based on the well-being of the person who is made worst off. Thus, in this situation, her utility for each outcome is the lowest payoff received between her and Bob. Figure 2 shows Alice's *true* utility for each outcome.

	C	D
C	2	0
D	0	1

Figure 2: The Rawlsian utilities for each outcome

Alice is unsure about whether Bob is also fair in the Rawlsian sense. She believes he is fair with probability p, but selfish with probability $1 - p$, in which case he will play based on the standard Prisoners' Dilemma payoff matrix. The probability p is common knowledge, which means that Bob knows that Alice knows p, Alice knows that Bob knows that Alice knows p, and so on. For concreteness, the two possible payoff matrices of the game are depicted below, in Figures 3 and 4.

	C	D
C	2, 2	0, 0
D	0, 0	1, 1

Figure 3: Bob is fair, like Alice; probability p.

	C	D
C	2, 2	0, 3
D	0, 0	1, 1

Figure 4: Bob is selfish, probability $1 - p$.

1. Suppose that $p < \frac{1}{3}$. What is the only ex-ante Bayes-Nash equilibrium in this game? *Hint:* Start by reasoning about what Bob will do if he is selfish.
2. Now, suppose that $p = \frac{1}{3}$. Give a pure-strategy ex-ante Bayes-Nash equilibrium that is different from the one that you found in part 1.
3. Finally, suppose that $p > \frac{1}{3}$. Find a *mixed-strategy* ex-ante Bayes-Nash equilibrium. Prove that your solution is an equilibrium.
4. Compare your solutions to parts 1 and 3. Which equilibrium is preferable for Alice and the fair version of Bob?

2 *Collusion and Dishonesty in the Second-Price Auction*

In class, we argued that in the second-price auction, no individual bidder has an incentive to deviate from bidding truthfully (i.e., reporting their value), regardless of the other bidders' behaviors. But the story is not quite so simple if bidders can *collude*, meaning submit bids in coordinated fashion.

Consider a second-price auction in which all but a subset S of the bidders bid truthfully. The members of S attempt to collude to increase their collective utility. State necessary and sufficient conditions on the valuations of the bidders in S relative to the others such that they can increase their collective utility via non-truthful bidding. Argue for the correctness of your conditions.

3 *The i.i.d. Assumption and First-Price Auctions*

The unique symmetric equilibrium strategy in a first-price auction where valuations are drawn i.i.d. from a distribution F is given by:

$$b_i(v_i) = v_i - \frac{\int_0^{v_i} F(x)^{n-1}\,\mathrm{d}x}{F(v_i)^{n-1}}. \tag{1}$$

Bids are "shaded" valuations at this equilibrium, meaning the former are less than the latter. Note also that this bid function is monotonically non-decreasing in valuation, so that a bidder with the highest valuation always wins the auction. In economic parlance, such an auction is called **efficient**.

As already noted, this equilibrium was derived under an important statistical assumption, namely that bidders' valuations are *independent and identically distributed* (i.i.d.). That is, they are all drawn from the same distribution (i.e., $F = F_i$, for all bidders i), and no one bidder's valuation depends on another's (independence).

Show that the i.i.d. condition is necessary to guarantee (economic) efficiency in first-price auctions by constructing an example in which valuations are not drawn i.i.d. and the outcome is not efficient. In your example, different bidders' valuations can be drawn from *different* distributions, or they can exhibit dependencies, or both. You need not derive an equilibrium; you need only argue that reasonable bidding strategies lead to an inefficient outcome.

4 *The Third-Price Auction*

The first- and second-price auctions aren't the only sealed-bid auctions to yield equivalent expected revenue. We call an auction in which the winner pays the third-highest bid a **third-price auction**. Whereas in a first-price auction, bidders shade their values at equilibrium, and in a second-price auction, bidders bid their true values at equilibrium, in a third-price auction, every bidder bids *higher* than their value at the symmetric Bayes-Nash equilibrium.

In answering the following questions, assume values are drawn i.i.d. from the uniform distribution on $[0, 1]$.

1. Prove that a bid of $b_i(v_i) = \left(\frac{n-1}{n-2}\right) v_i$ by every bidder i is a symmetric Bayes-Nash equilibrium of the third-price auction.
2. Show that the expected revenue R_3 in the third-price auction is
$$R_3 = \frac{n-1}{n+1}.$$
3. What is the symmetric Bayes-Nash equilibrium strategy in a kth price auction? (You need only state how each bidder bids; you need not provide a detailed analysis.)

5 *The All-Pay Auction (Just for Fun!)*

We introduce another auction called the **all-pay auction**. Here, the good is awarded to the highest bidder, but rather than just the winner pay, *every* bidder must pay their bid.

In answering the following questions, assume values are drawn i.i.d. from the uniform distribution on $[0, 1]$.

1. Prove that a bid of $b_i(v_i) = \left(\frac{n-1}{n}\right) v_i^n$ by every bidder i is a symmetric Bayes-Nash Equilibrium.
2. Show that the expected revenue R_{ALL} for the all-pay auction is
$$R_{\text{ALL}} = \frac{n-1}{n+1}.$$

6 *First-Price Auction Strategy*

Suppose you enter a first-price auction with n bidders. Once again, valuations are i.i.d. on the uniform distribution on $[0, 1]$. However, this time bidders don't necessarily abide by the equilibrium. Instead, they have all chosen a bid function $\theta(n)$ that is linear in their valuation. So, for each bidder j other than you,

$$b_j(v_j) = \theta(n) v_j.$$

Note that we use $\theta(n)$ and not $\theta_j(n)$ because every bidder uses the same function $\theta(n)$. Also note that there is no constant term, which means that if a bidder values the good at 0, she will bid 0.

Imagine you are bidder i. Derive your optimal (i.e., expected utility-maximizing) bid b_i in terms of v_i, n, and $\theta(n)$. How does your optimal bid relate to the equilibrium of such a first-price auction (i.e., n bidders, valuations drawn i.i.d. on $U[0, 1]$)?