Partner 1
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Homework 4

CSCI 1550 / 2540
March 20th, 2025

Due: April 10th, 2025
Remember to show your work for each problem to receive full credit.

Problem 1 (30 points)

1. Let $X_1, X_2, \ldots,$ be a sequence of independent exponential random variables, each with mean 1. Given a positive real number k, let N be defined by

$$N = \min \left\{ n : \sum_{i=1}^{n} X_i > k \right\}.$$

That is, N is the smallest number for which the sum of the first N of the X_i is larger than k. Use Wald's inequality to determine $E[N]$.

Solution: It is easy to see that $\mathbb{E}(N)$ is finite. Therefore, we can apply Wald's inequality, and obtain that

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\mathbb{E}(X) = \mathbb{E}(N)$$

Let $y = k - \sum_{i=1}^{N-1} X_i > 0$. Since the sum of $X_1, \ldots, X_N$ is greater than k, we have that $X_N = y + \tilde{X}$, where $\tilde{X}$ represents how much the sum is greater than k, i.e. $\tilde{X} = \sum_{i=1}^{N} X_i - k$. Conditioning on the fact that X_N is greater than y, we have that $\tilde{X}$ is still distributed as a exponential random variable with mean 1 due to the memoryless property of exponentials. Therefore, we have that

$$\mathbb{E}(N) = \mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}\left(\sum_{i=1}^{N-1} X_i + y + \tilde{X}\right) = k + \mathbb{E}(\tilde{X}) = k + 1$$

2. Let $X_1, X_2, \ldots,$ be a sequence of independent uniform random variables on the interval $(0, 1)$. Given a positive real number k, with $0 < k < 1$, let N be defined by

$$N = \min \left\{ n : \prod_{i=1}^{n} X_i < k \right\}.$$

That is, N is the smallest number for which the product of the first N of the X_i is smaller than k. Determine $E[N]$. (Hint: Prove that $\log 1/X_i$ has an exponential distribution.)

Solution: We can rewrite N as

$$N = \min\left\{n : \prod_{i=1}^{n} X_i < k\right\} = \min\left\{n : \sum_{i=1}^{n} \ln(1/X_i) > \ln(1/k)\right\}$$

This equality immediately follows by taking the logarithm of both sides of the equation. Consider $Y_i = \ln(1/X_i)$. We have that

$$\Pr(Y_i \le x) = \Pr(\ln(1/X_i) \le x) = \Pr(X_i \ge e^{-x}) = 1 - e^{-x}$$

Therefore Y_i is distributed as a exponential random variable with mean 1. By using the result of (a), we have that

$$\mathbb{E}(N) = \ln(1/k) + 1$$

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Problem 2 (40 points)

A random graph $G_{n,m}$ has n vertices and m edges. The m edges are chosen uniformly at random (without repetition) among all the possible $\binom{n}{2}$ edges. Consider a random graph $G_{n,m}$, where $m = cn$ for some constant $c > 0$. Let X be the number of isolated vertices (i.e., vertices of degree 0).

1. Compute $E(X)$

   Solution: Consider a vertex v. The number of graphs where v is disconnected is equal to $\binom{\binom{n-1}{2}}{m}$. The total number of random graphs is equal to $\binom{\binom{n}{2}}{m}$. Therefore, the probability of v being isolated is:

$$\frac{\binom{\binom{n-1}{2}}{m}}{\binom{\binom{n}{2}}{m}}$$

   By linearity of expectation, the expected number of isolated vertices is

$$n\frac{\binom{\binom{n-1}{2}}{m}}{\binom{\binom{n}{2}}{m}}$$

2. For any $\lambda > 0$, show that $\Pr(|X - E(X)| \geq 2\lambda\sqrt{cn}) \leq 2e^{-\lambda^2/2}$ (*Hint*: Use a martingale that reveals the locations of the edges in the graph, one at a time).

   Solution: Consider the edges $E_1, \ldots, E_m$ of the random graph G, Let $f(G)$ denote the number of the isolated vertices in G. We construct the edge-exposure martingale $Z_i = [f(G)|E_1, \ldots, E_i]$, for $i = 1, \ldots, m$. Note that $Z_m = X$ is equal to the number of isolated vertices. Let $Z_0 = [f(G)] = \exp(X)$ be the expected number of isolated vertices. Now, we can observe that for any $i = 1, \ldots, m$, we have that $|Z_i - Z_{i-1}| \leq 2$, as revealing an edge can decrease the number of non-isolated vertices by at most 2 (why? an edge can only connect to at most 2 unvisited vertices, then these 2 vertices cannot be isolated anymore). By Azuma-Hoeffding inequality, we have that:

$$\begin{aligned}
\Pr(|X - E(X)| \geq 2\lambda\sqrt{cn}) &= \Pr(|Z_m - Z_0| \geq 2\lambda\sqrt{cn}) \\
&\leq 2\exp(-4\lambda^2 cn/(2\sum_{i=1}^{cn} 4)) \\
&= 2\exp(-\lambda^2/2)
\end{aligned}$$

Problem 3 (25 points)

Consider a random walk on the infinite two dimension integer grid:

$$G = \{(x,y) \mid x \in \{-\infty, \infty\},\ y \in \{-\infty, \infty\}\}.$$

The random walk starts at $(0,0)$, and if the walk is at (x_t, y_t) at time t, then with equal probabilities the walk moves to one of the adjacent nodes $(x_t - 1, y_t)$, $(x_t, y_t - 1)$, $(x_t + 1, y)$, or $(x_t, y_t + 1)$. I.e.

$$Pr((x_{t+1}, y_{t+1})) \mid (x_t, y_t)) = \begin{cases} 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t - 1, y_t) \\ 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t + 1, y_t) \\ 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t, y_t - 1) \\ 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t, y_t + 1) \end{cases}$$

Prove that for $\lambda > 0$

$$Pr\left(|x_t + y_t| \geq \lambda\sqrt{t}\right) \leq 2e^{-\lambda^2/2}.$$

Solution: Let $z_t = x_t + y_t$. We first show that z_t is a martingale with respect to itself. Clearly $\mathbb{E}[|z_t|] \leq t$. Furthermore,

$$\begin{aligned} \mathbb{E}[z_{t+1}|z_t] &= (z_t + 1)Pr(z_{t+1} = z_t + 1|z_t) + (z_t - 1)Pr(z_{t+1} = z_t - 1|z_t) \\ &= (z_t + 1)\frac{1}{2} + (z_t - 1)\frac{1}{2} \\ &= z_t \end{aligned}$$

(there is an abuse of notation: z_t represents both a random variable and some fixed integer) Also notice that $|z_{t+1} - z_t| \leq 1$, which allows us to use Corollary 13.5 on the book. $\forall \lambda > 0$ and $t \geq 0$, we have

$$\begin{aligned} Pr(|z_t - z_0| \geq \lambda\sqrt{t}) &\leq 2e^{-\lambda^2/2} \\ Pr(|x_t + y_t| \geq \lambda\sqrt{t}) &\leq 2e^{-\lambda^2/2} \end{aligned}$$

where we have the last inequality because $z_0 = 0$.

Problem 4 (25 points)

Let $f(X_1, X_2, \ldots, X_n)$ satisfy the Lipschitz condition so that, for any i and any values $x_1, \ldots, x_n$ and y_i,

$$|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \leq c.$$

We set

$$Z_0 = \mathbb{E}(f(X_1, \ldots, X_n))$$

and

$$Z_i = \mathbb{E}(f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i).$$

Give an example to show that, if the X_i are not independent, then it is possible that $|Z_i - Z_{i-1}| > c$ for some i.

Solution: Let X_1 be a random variable that takes value 0 with probability 1/2 and 1 with probability 1/2. Let $X_2 = X_3 = \ldots = X_n = X_1$. Let

$$f(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i$$

Because each X_i is a 0/1 -random variable, f satisfies the Lipschitz condition with Lipschitz constant $c = 1$. Because $f(X_1, \ldots, X_n)$ takes value 0 with probability 1/2 and value n with probability 1/2 we have $Z_0 = E(f(X_1, \ldots, X_n)) = n/2$. But Z_1 is either 0 or n, because all the X_i are determined by the value of X_1. So, for $n > 2$, we have $|Z_1 - Z_0| = n/2 > c$.

Problem 5 (30 points)

Consider a bin with $N > 1$ balls. The balls are either black or white. Let $X_0 = \frac{m}{N} < 1$ be the fraction of black balls in the bin at time 0. Let X_i be the fraction of black balls at time i. At step $i \geq 1$ one ball, chosen uniformly at random, is replace with a new ball. With probability X_i the new ball is black, otherwise it is white. All random choices are independent.

Consider the stopping time $\tau := \inf_i\{X_i \in \{0, 1\}\}$. That is, the process stops when all balls have the same color.

(a) Show that $X_1, X_2, \ldots$ is a martingale with respect to itself.

Solution: We see that

$$\mathbb{P}\left(X_{n+1} = \beta \mid X_n = \alpha\right) = \begin{cases} \alpha(1-\alpha), \beta = \alpha - 1/N \\ \alpha^2 + (1-\alpha)^2, \beta = \alpha, \\ (1-\alpha)\alpha, \beta = \alpha + 1/N \end{cases} \qquad \text{when } \alpha \in (0,1),$$

and

$$\mathbb{P}\left(X_{n+1} = \beta \mid X_n = \alpha\right) = 1_\alpha(\beta)$$

Checking manually, we see that $\mathbb{E}[X_{n+1}|X_n] = X_n$. As $X_n \in [0,1]$, we conclude that $\{X_n\}$ is a martingale.

(b) Show that $E[\tau] < \infty$.
Hint: Show that at any step there is probability $\geq (\frac{1}{2N})^{N/2}$ to terminate in the next $N/2$ steps. Conclude that $E[\tau] \leq (2N)^{N/2} \cdot N/2$.

Solution: Let us divide the process into time blocks of size $N/2$. Regardless of the value of X_n $\min(X_n, 1 - X_n) \leq 1/2$. We add an additional ball of a given color by choosing distinct colors for the old and new ball. Before stopping, this occurs with probability at least $\frac{m(N-m)}{N} \geq \frac{1}{N^2}$. Choosing $N/2$ balls of a certain color will stop the process, and by independence of our choices, we will terminate with probability at least $\frac{1}{N^2}^{N/2} = \frac{1}{N^N}$ in each block. Then we will not have terminated after i blocks with probability at most $(1 - \frac{1}{N^N})^i$. Then taking the expectation over blocks:

$$\mathbb{E}[\tau] = \sum_{i=0} \underbrace{P(\tau \geq i)}_{\text{why?}} \leq \sum_{i=0} \frac{N}{2} P(\tau \geq \frac{iN}{2}) \leq \sum_{i=0} \frac{N}{2}\left(1 - \frac{1}{N^N}\right)^i = \frac{N}{2}\frac{1}{1 - \left(1 - \frac{1}{N^N}\right)} = \frac{N}{2} * N^N$$

which is finite.

(c) Calculate $\mathbb{P}(X_\tau = 1)$

Solution: By part (b), we can apply the Stopping Theorem, so that

$$P(X_\tau = 1) = \mathbb{E}[X_\tau] = \mathbb{E}[X_0] = \frac{m}{N}$$