CSCI 1510

This Lecture:

- Factoring / RSA & DLOG / CDH / DDH Assumptions (continued)
- Key Exchange Definition & Construction
- Public-key Encryption Definitions
- ElGamal / RSA Encryption
- Trapdoor Permutations

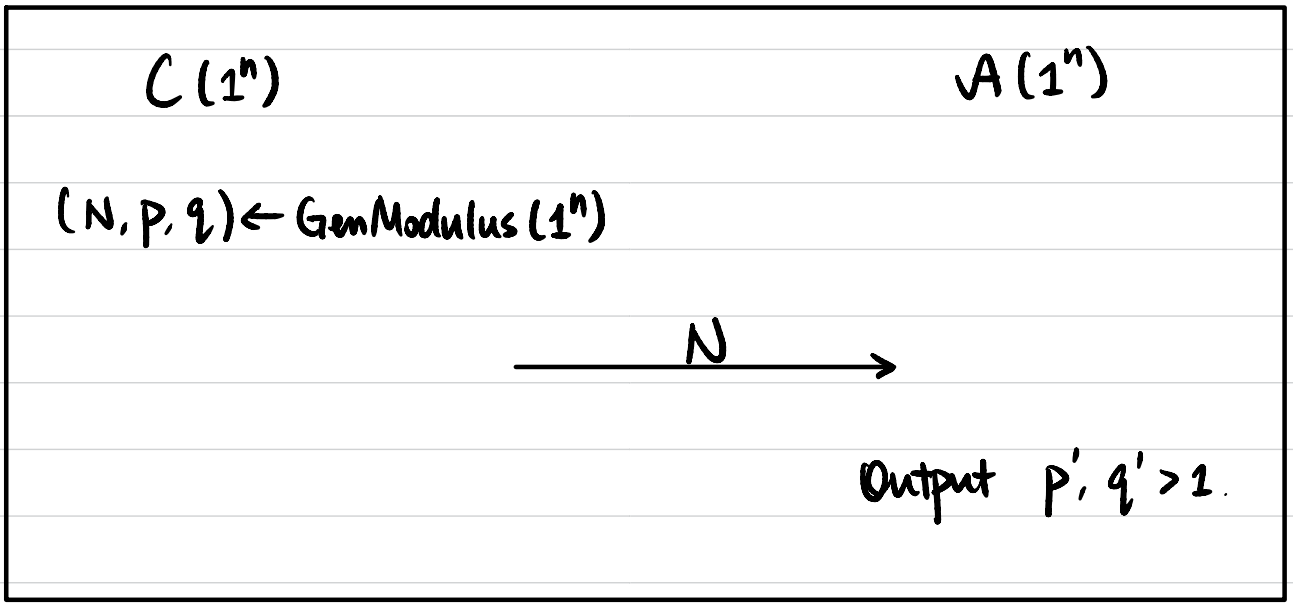
Factoring Assumption

GenModulus (1^n): PPT algorithm, generates (N, p, q)

p, q: n-bit primes, $p \neq q$. $N = p \cdot q$

Def Factoring is hard relative to GenModulus if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[p' \cdot q' = N] \leq \varepsilon(n)$.



Factoring $\Rightarrow$ OWF (GenModulus)

RSA Assumption

GenModulus(1^n): generates (N, p, q). p, q: n-bit primes, $p \neq q$. $N = p \cdot q$

GenRSA(1^n):

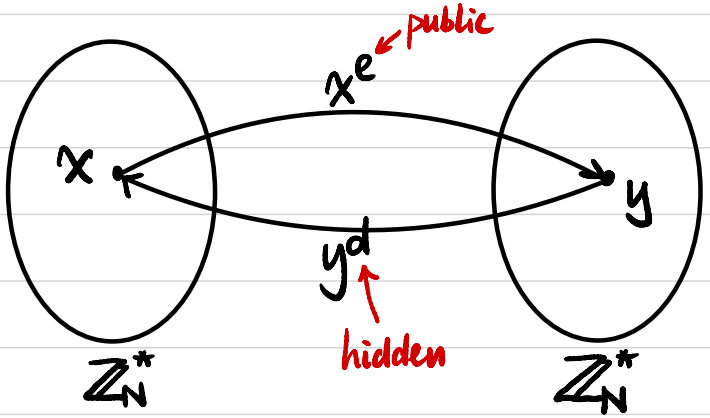
$(N, p, q) \leftarrow \text{GenModulus}(1^n)$

$\phi(N) := (p-1)(q-1)$ ← prime

Choose $e > 1$ s.t. $\gcd(e, \phi(N)) = 1$

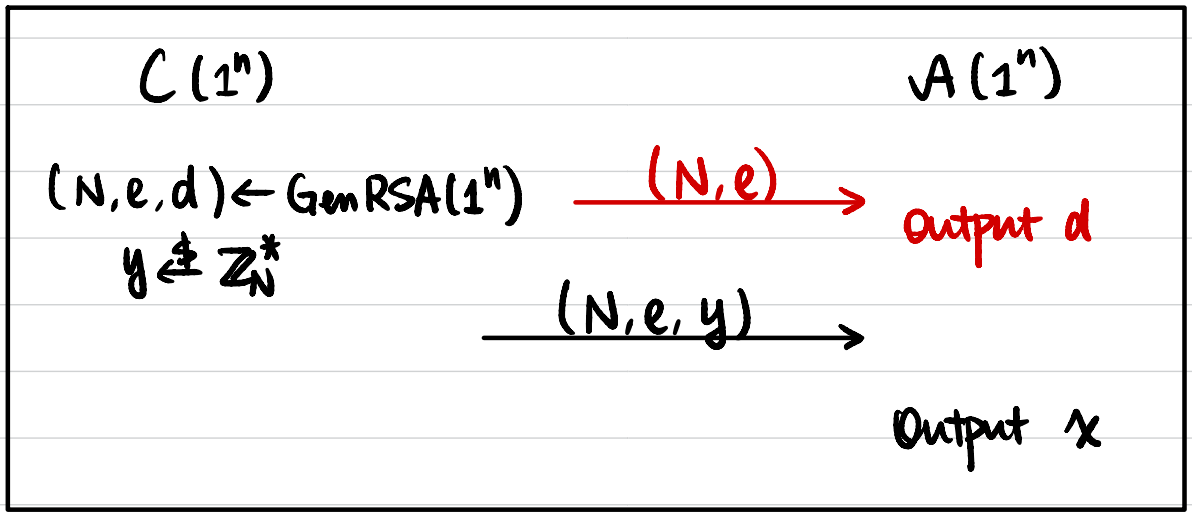
Compute $d := e^{-1} \bmod \phi(N)$

Output (N, e, d)



Def The RSA problem is hard relative to GenRSA if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[x^e = y \bmod N] \leq \varepsilon(n)$.



RSA $\overset{?}{\Rightarrow}$ Factoring

Discrete-Log Assumption

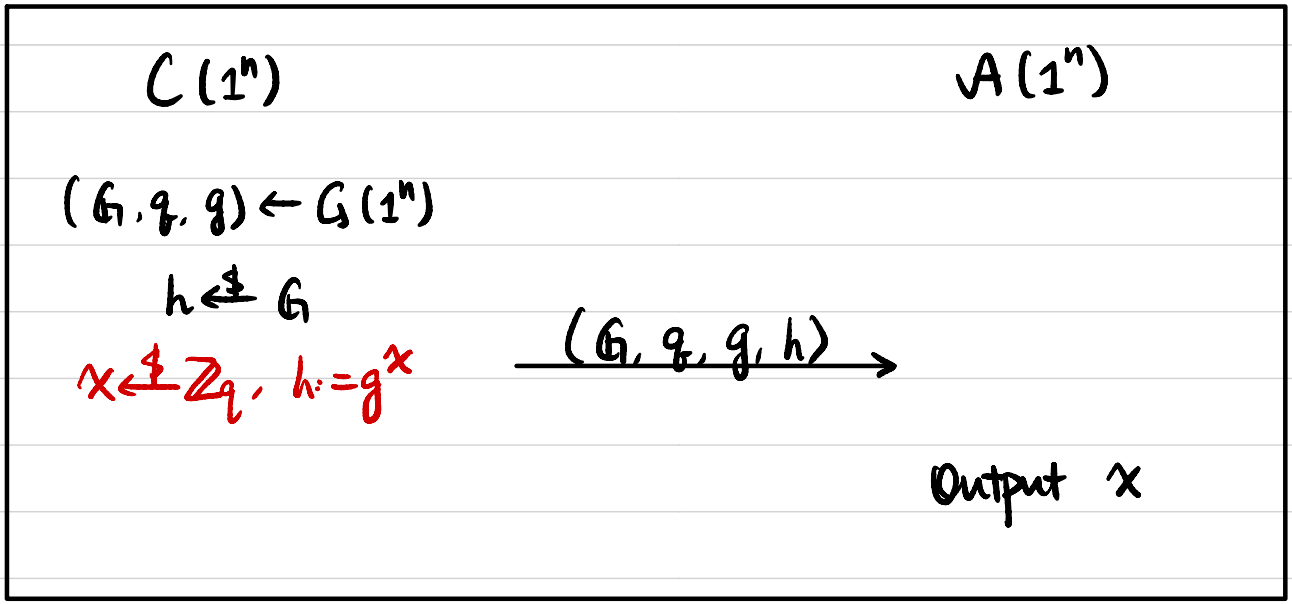
$G(1^n)$: PPT algorithm, generates (G, q, g)

$G = \{g^0, g^1, g^2, \cdots, g^{q-1}\}$

description of a cyclic group G of order q (↑ n-bit integer) with generator g.

Def Discrete-Log (DLOG) is hard relative to G if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[g^x = h] \leq \varepsilon(n)$.



$g^x \xrightarrow{?} x$

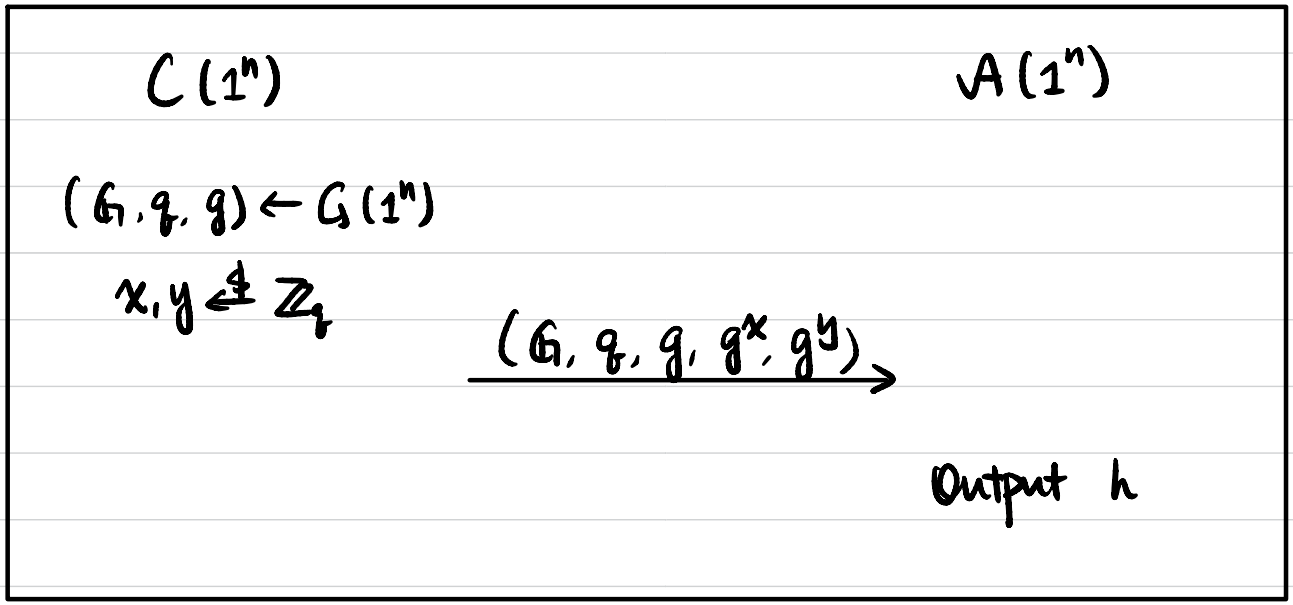
DLOG ⇒ CRHF ⇒ OWF

Computational Diffie-Hellman (CDH) Assumption

$G(1^n)$: PPT algorithm, generates (G, q, g)

Def CDH is hard relative to G if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[h = g^{xy}] \leq \varepsilon(n)$.



$(g^x, g^y) \xrightarrow{?} g^{xy}$

CDH $\Rightarrow$ DLOG

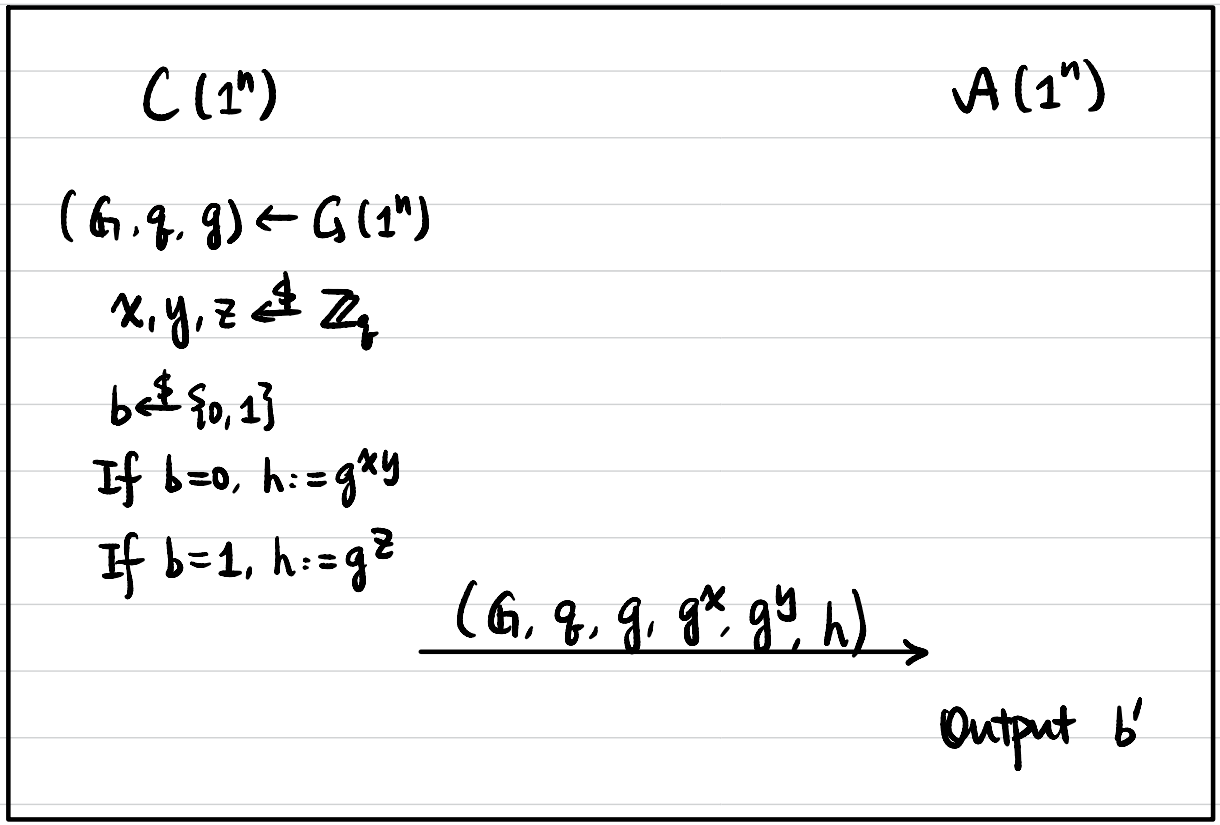
break $\exists$ PPT A

Decisional Diffie-Hellman (DDH) Assumption

$G(1^n)$: PPT algorithm, generates (G, q, g)

Def DDH is hard relative to G if

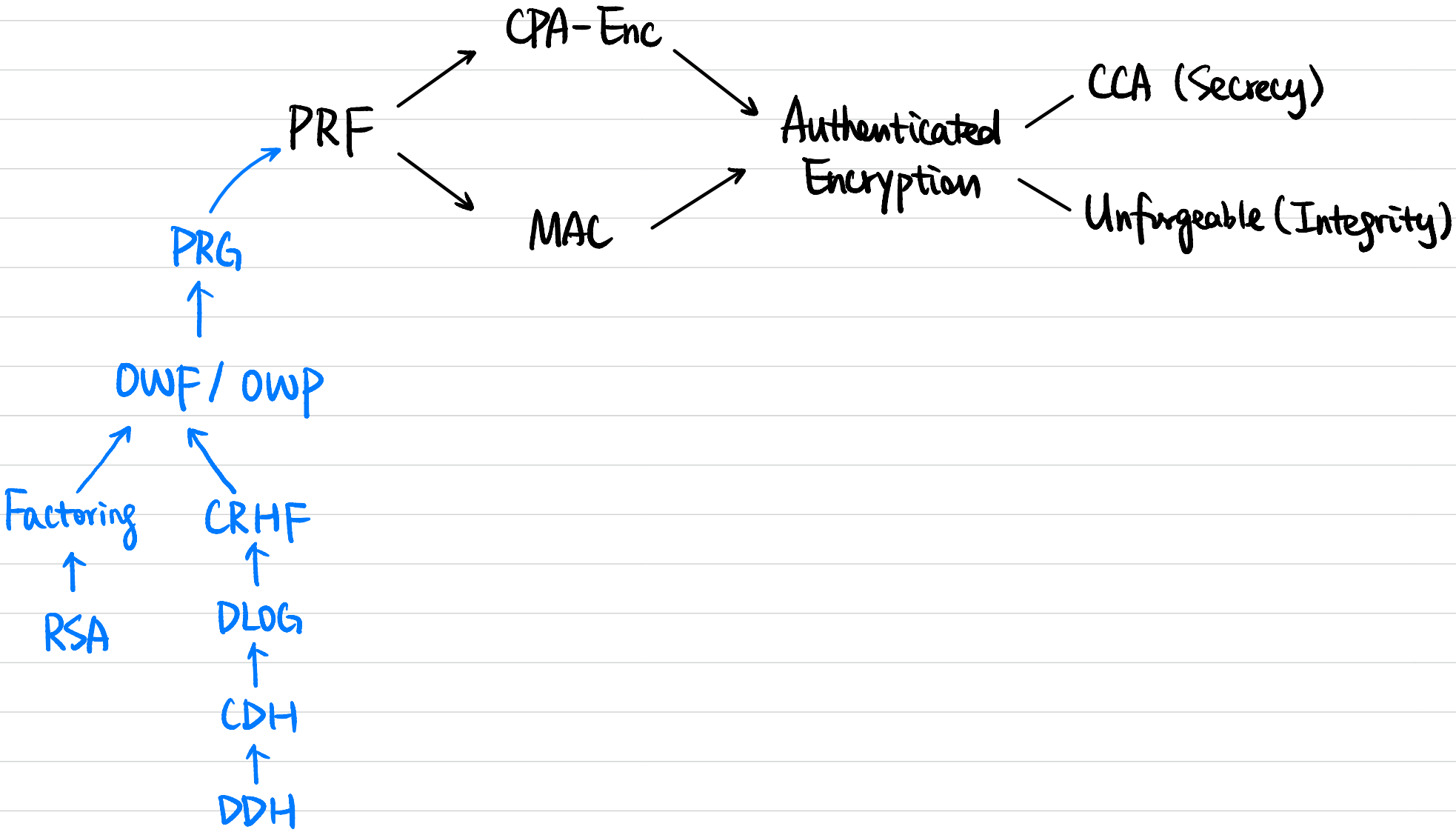
$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[b = b'] \leq \frac{1}{2} + \varepsilon(n)$.



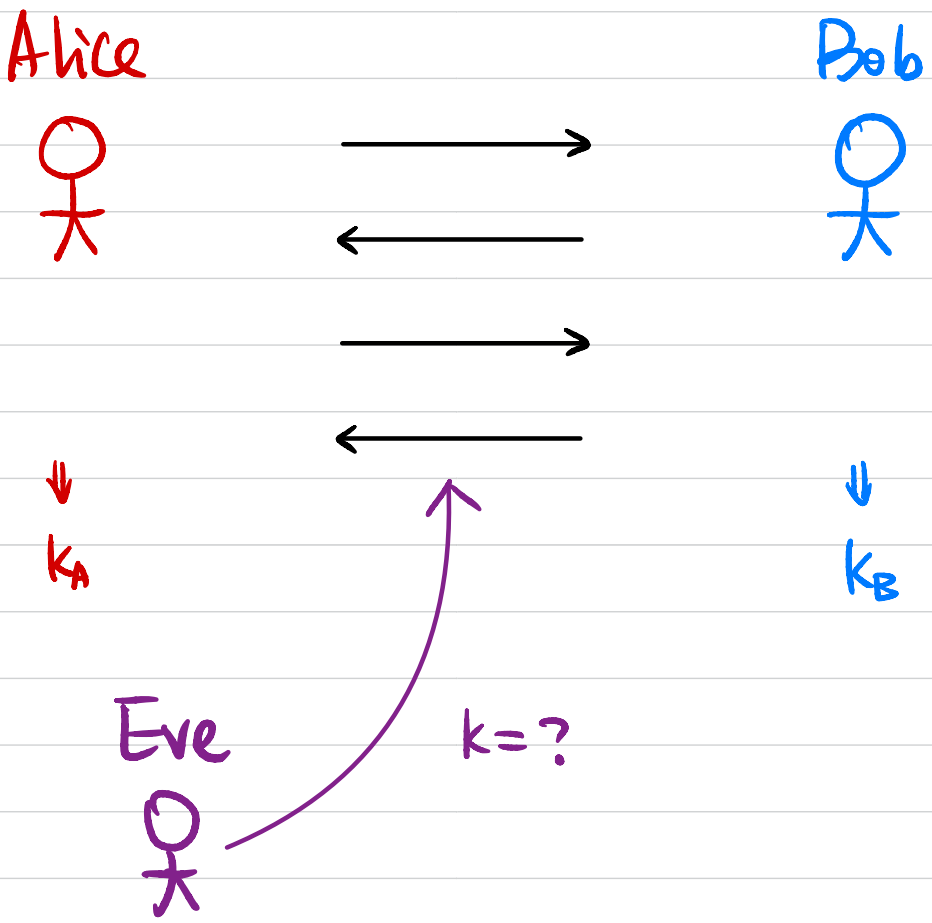
$(g^x, g^y, g^{xy}) \overset{c}{\simeq} (g^x, g^y, g^z)$

DDH $\Rightarrow$ CDH

break $\exists$ PPT A


CPA-Enc
PRF
MAC
Authenticated Encryption
CCA (Secrecy)
Unforgeable (Integrity)
PRG
OWF / OWP
Factoring
CRHF
RSA
DLOG
CDH
DDH

Key Exchange



- **Correctness:** $k = k_A = k_B$
- **Security (Informally):** Eve listening on the channel shouldn't be able to guess k.

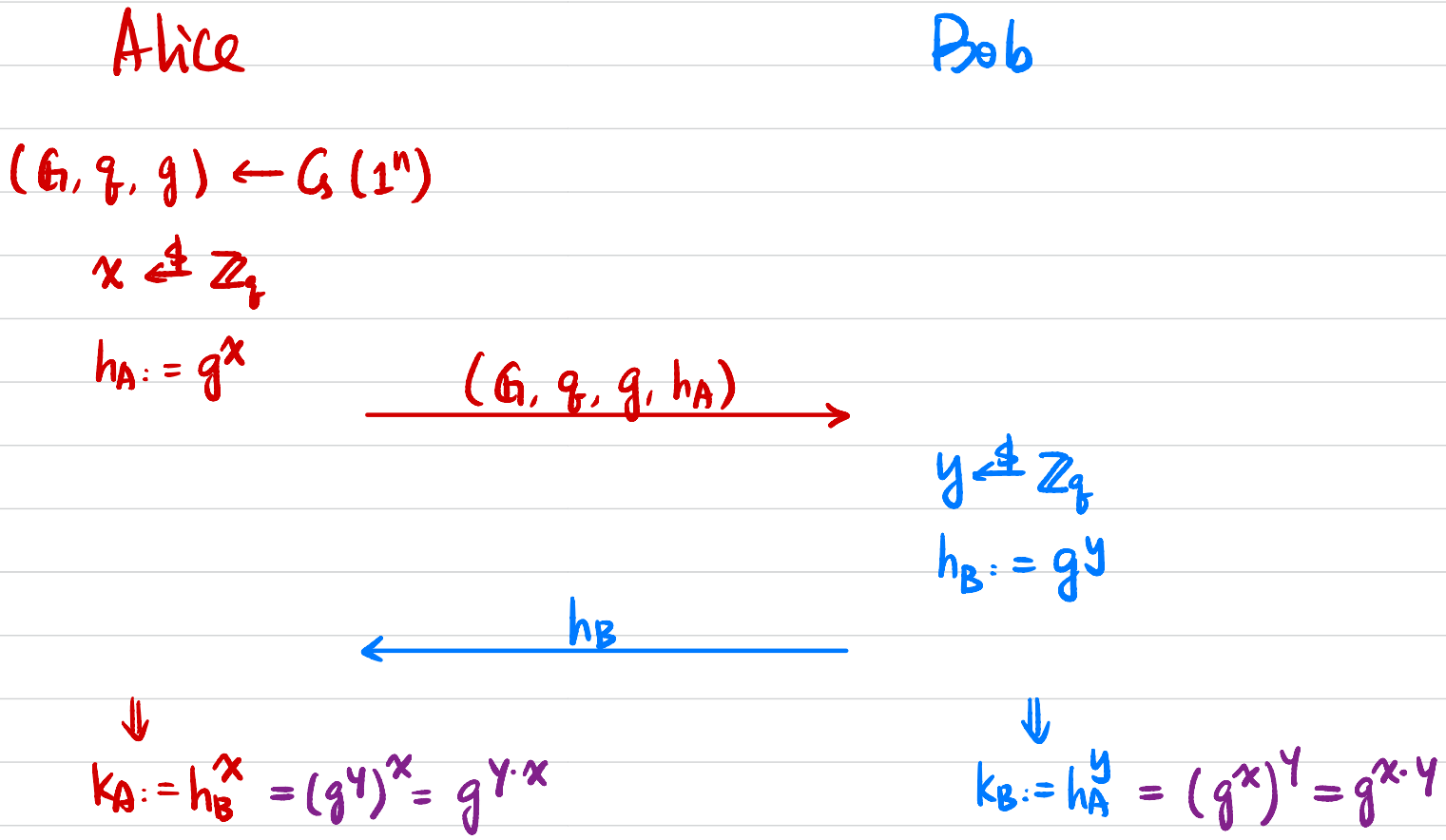
Key Exchange: Security

Def A key exchange protocol Π is secure if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[b = b'] \leq \frac{1}{2} + \varepsilon(n)$.

$C(1^n)$	$A(1^n)$
Two parties holding 1^n execute Π.	
$\Rightarrow$ transcript T containing all the messages & a key k output by each party.	
$b \xleftarrow{\$} \{0,1\}$	
If $b=0$, $\hat{k} := k$	
If $b=1$, $\hat{k} \xleftarrow{\$} \{0,1\}^n$	
$\xrightarrow{(T, \hat{k})}$	
	output b'

Diffie-Hellman Key Exchange

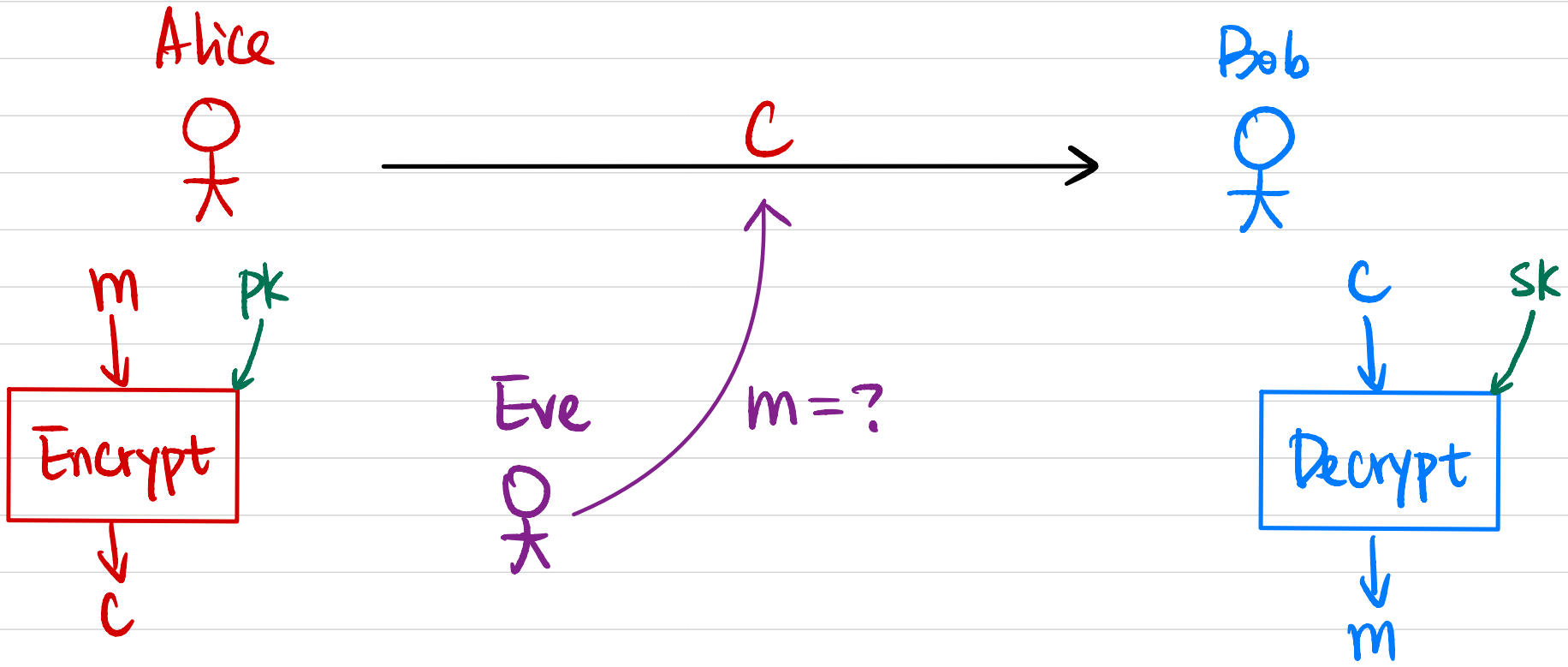


Thm If DDH is hard relative to G, then this is a secure key exchange protocol.

$b=0$: $T = (G, q, g, h_A, h_B)$, $\hat{k} = g^{xy}$

$b=1$: $T = (G, q, g, h_A, h_B)$, $\hat{k} = g^z$

Public-Key Encryption



Public-Key Encryption

- **Syntax:**

  A public-key encryption (PKE) scheme is defined by PPT algorithms (Gen, Enc, Dec):

  $(pk, sk) \leftarrow Gen(1^n)$

  $C \leftarrow Enc_{pk}(m)$

  $m/\perp := Dec_{sk}(c)$

- **Correctness:** $\forall (pk, sk)$ output by $Gen(1^n)$, $\forall m \ (\in M_{pk})$.

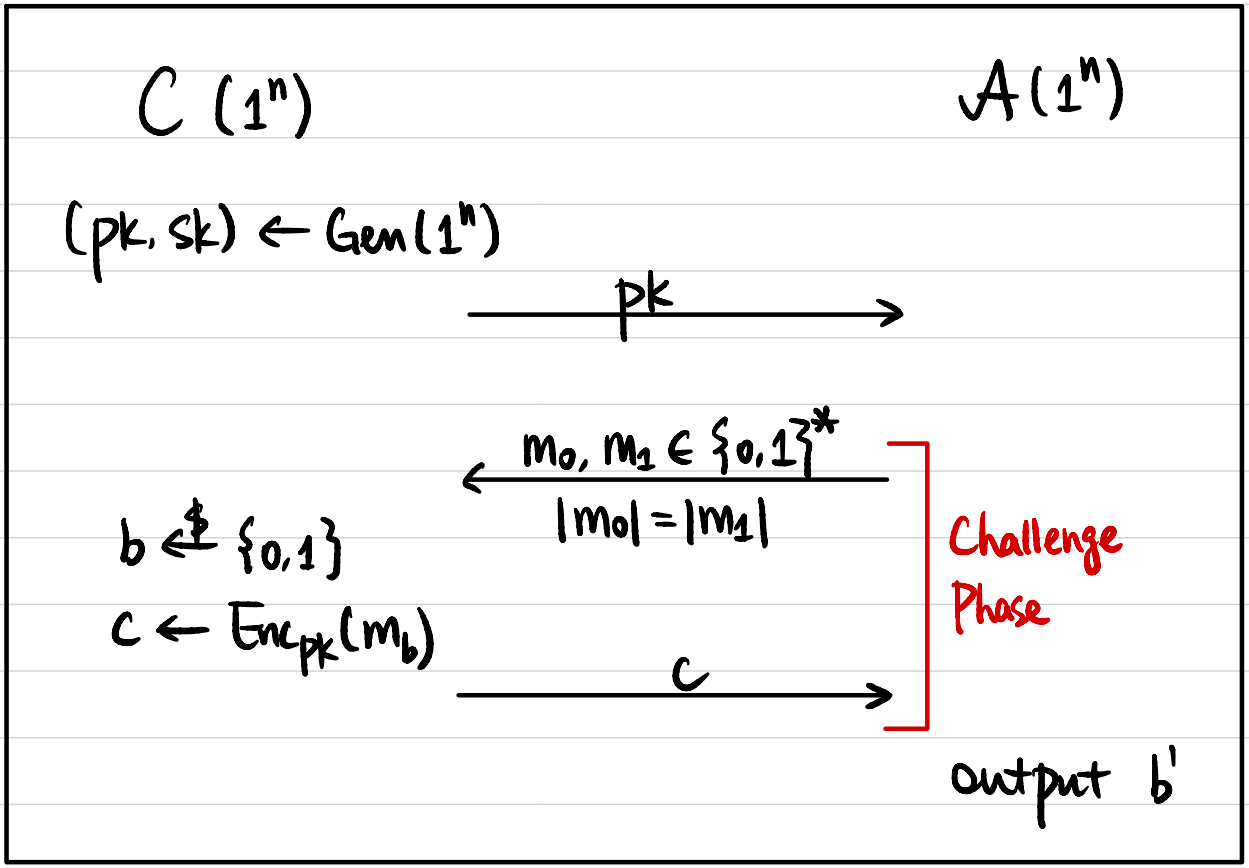
  $$Dec_{sk}(Enc_{pk}(m)) = m.$$

- **Security:** Semantic / CPA / CCA?

Semantic Security

Def A public-key encryption scheme (Gen, Enc, Dec) is **semantically secure** (= CPA) if $\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t.

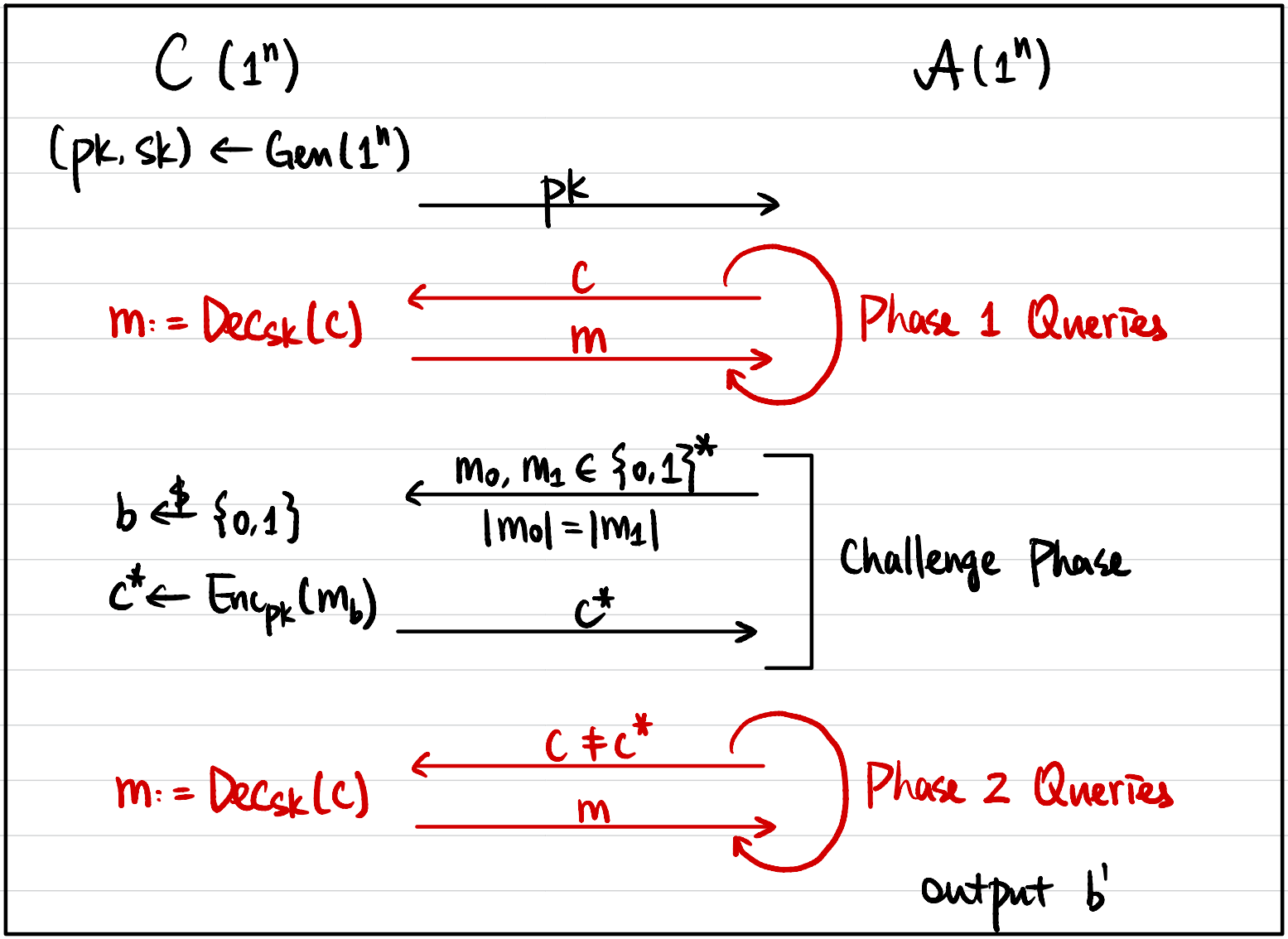
$$\Pr[b = b'] \leq \frac{1}{2} + \varepsilon(n)$$



CPA Security ?

Chosen Ciphertext Attack (CCA) Security

Def A public-key encryption scheme (Gen, Enc, Dec) is **CCA-secure** if $\forall$ PPT $\mathcal{A}$, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[b = b'] \leq \frac{1}{2} + \varepsilon(n)$



ElGamal Encryption

- $Gen(1^n)$:
  $(G, q, g) \leftarrow G(1^n)$
  $x \xleftarrow{\$} \mathbb{Z}_q$, $h := g^x$
  $pk := (G, q, g, h)$
  $sk := x$

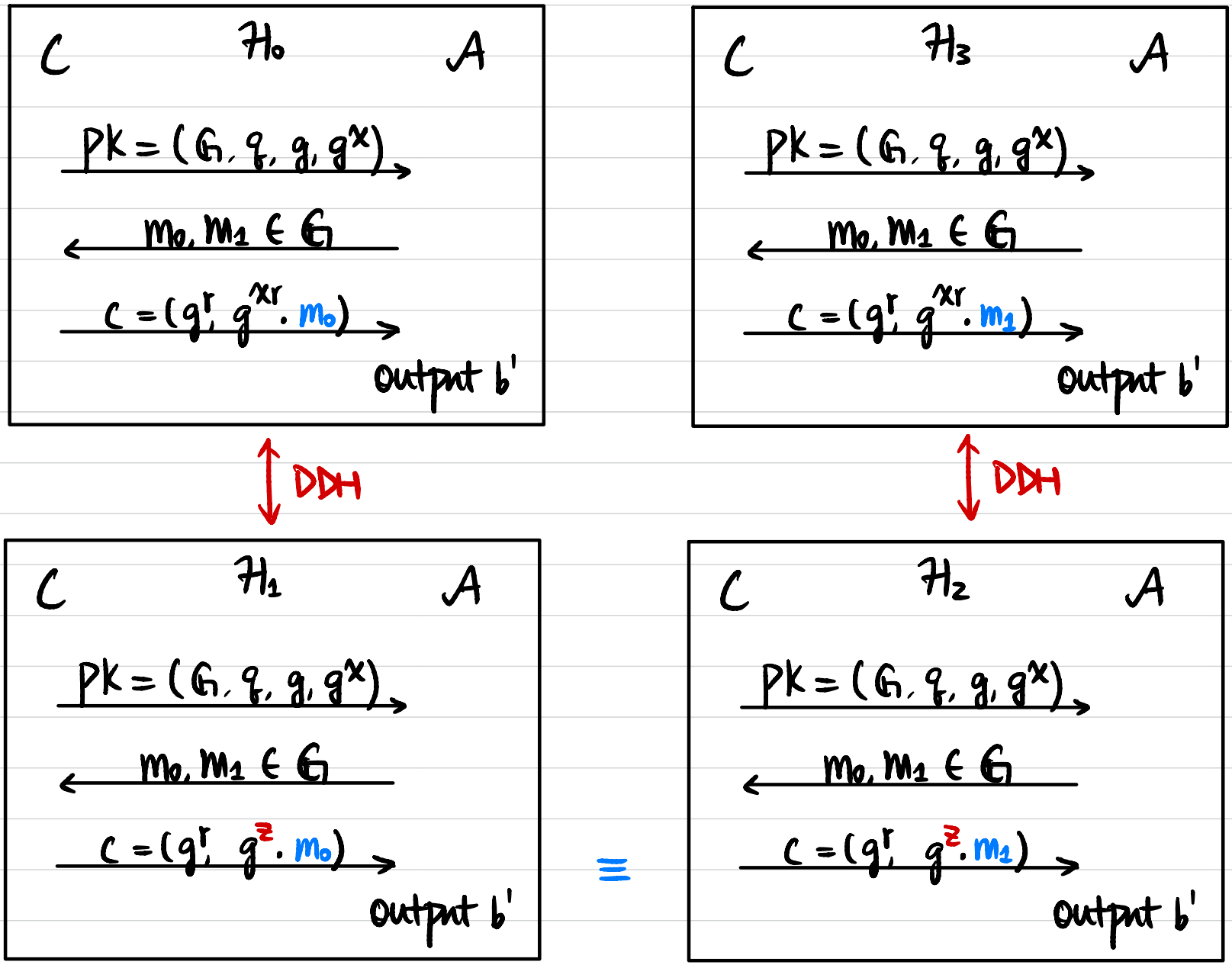
- $Enc_{pk}(m)$: $m \in G$
  $r \xleftarrow{\$} \mathbb{Z}_q$ ← g^{xr}
  $c := (g^r, h^r \cdot m)$

- $Dec_{sk}(c)$: $c = (c_1, c_2)$
  $c_1^x \rightarrow g^{xr}$
  $m := c_2 \cdot (c_1^x)^{-1}$

Thm If DDH is hard relative to G, then ElGamal encryption is CPA-secure.

Thm If DDH is hard relative to G, then ElGamal encryption is CPA-secure.

Proof Sketch



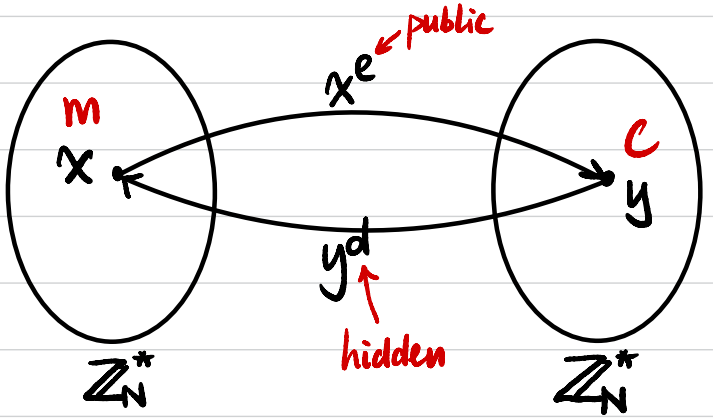
RSA-based Encryption

Plain RSA Encryption:

- $Gen(1^n)$:
  $(N, e, d) \leftarrow GenRSA(1^n)$
  $pk := (N, e)$
  $sk := (N, d)$

- $Enc_{pk}(m)$: $m \in \mathbb{Z}_N^*$
  $c := m^e \bmod N$

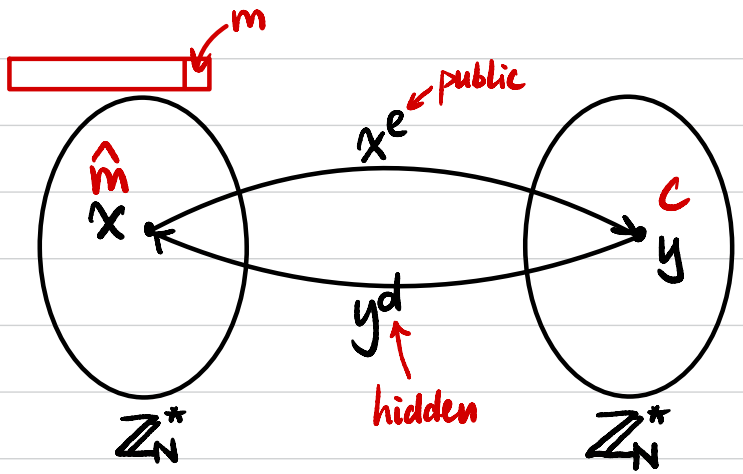
- $Dec_{sk}(c)$: $m := c^d \bmod N$



Is it CPA-secure? No!

RSA-based Encryption

Padded RSA Encryption:



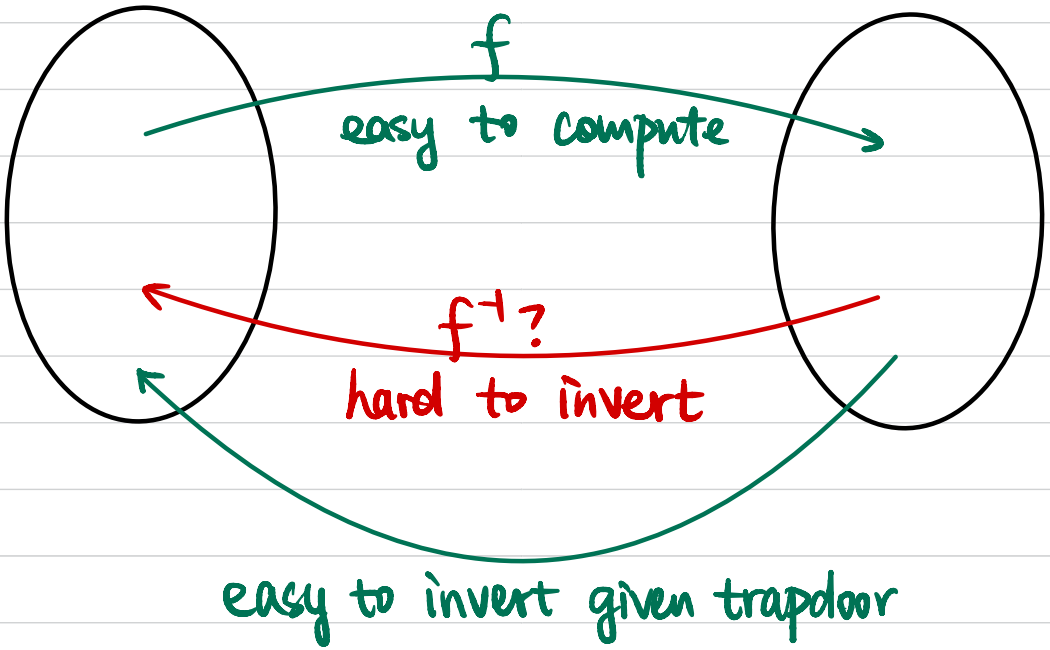
- $\text{Gen}(1^n)$:
  $(N, e, d) \leftarrow \text{GenRSA}(1^n)$
  $pk := (N, e)$
  $sk := (N, d)$

- $\text{Enc}_{pk}(m)$: $m \in \{0,1\}$
  least significant bit
  $\hat{m} \xleftarrow{\$} \mathbb{Z}_N^*$ s.t. $\text{lsb}(\hat{m}) = m$
  $c := \hat{m}^e \bmod N$

- $\text{Dec}_{sk}(c)$: $\hat{m} := c^d \bmod N$
  $m := \text{lsb}(\hat{m})$

Thm If the RSA problem is hard relative to GenRSA, then this encryption scheme is CPA-secure.

Trapdoor Permutation



Trapdoor Permutation

Def A family $F = \{f_i : D_i \to R_i\}_{i \in I}$ is a **trapdoor permutation** if

① permutation: $\forall i \in I$, f_i is a permutation (bijection) $i = (N, e)$

② easy to sample a function: $(i, t) \leftarrow Gen(1^n)$. $f_i(x) = x^e \bmod N$

③ easy to sample an input: $x \leftarrow Sample(i \in I)$. x uniform in D_i.

④ easy to compute f_i: $f_i(x)$ poly-time computable $\forall i \in I, x \in D_i$.

⑤ hard to invert f_i: $\forall PPT\ A$, $\exists$ negligible function $\varepsilon(\cdot)$ s.t.

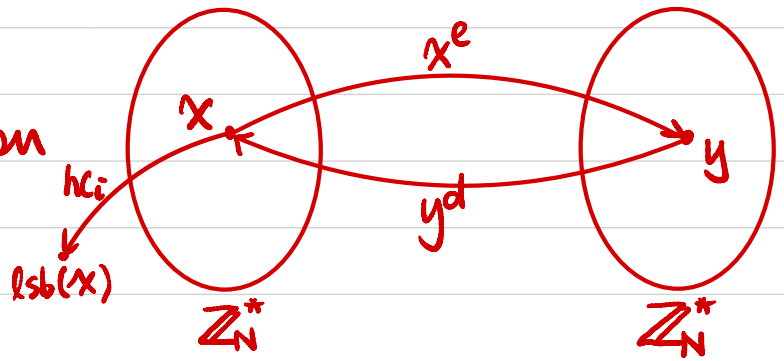
RSA Assumption

$$Pr\left[\begin{array}{l}(i,t) \leftarrow Gen(1^n), \\ x \leftarrow Sample(i) \\ y \leftarrow f_i(x) \\ z \leftarrow A(1^n, i, y)\end{array} : f_i(z) = y\right] \le \varepsilon(n).$$

⑥ easy to invert f_i with trapdoor: $Inv(i, t, f_i(x)) = x$ $(i, t) \leftarrow Gen(1^n)$, $x \in D_i$

(t = d)

Example: RSA trapdoor permutation

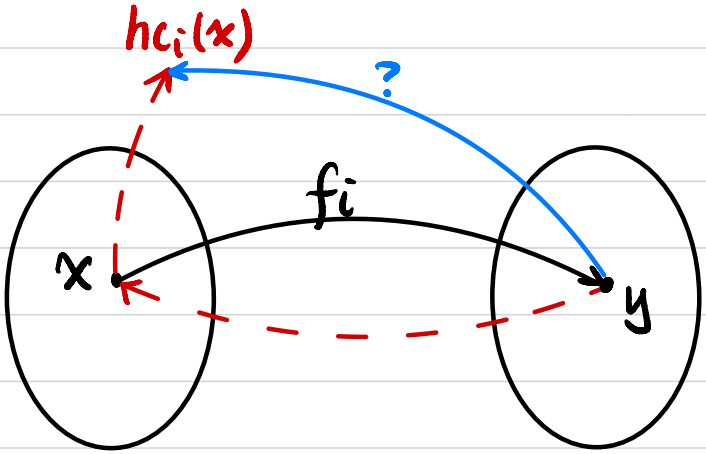


Hard-Core Predicate

Def Let $\Pi = (F, Gen, Inv)$ be a trapdoor permutation,
Let hc be a deterministic poly-time algorithm that, on input i & $x \in D_i$,
Outputs a single bit $hc_i(x)$.

hc is a hard-core predicate of Π if
$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t.

$$\Pr_{\substack{(i,t)\leftarrow Gen(1^n) \\ x \leftarrow D_i}} [A(i, f_i(x)) = hc_i(x)] \leq \frac{1}{2} + \varepsilon(n)$$



Thm Assume trapdoor permutation exists.
Then there exists a trapdoor permutation Π with a hard-core predicate hc of Π.

PKE from TDP

- $\text{Gen}(1^n)$:

  $(i, t) \leftarrow \text{Gen}(1^n)$

  $pk := i$

  $sk := t$

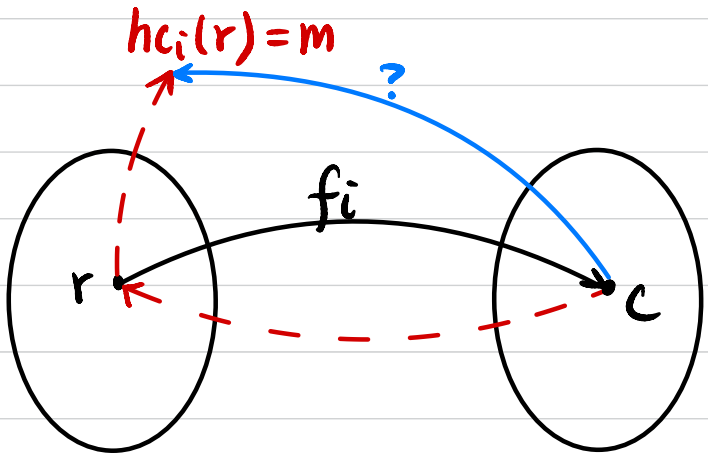
- $\text{Enc}_{pk}(m)$: $m \in \{0,1\}$

  $r \leftarrow D_i$ s.t. $hc_i(r) = m$

  $c := f_i(r)$

- $\text{Dec}_{sk}(c)$:

  $m := hc_i(\text{Inv}(i, t, c))$



Thm If $\Pi = (F, \text{Gen}, \text{Inv})$ be a trapdoor permutation with a hard-core predicate hc, then this encryption scheme is CPA-secure.