CSCI 1510

This Lecture:

- Basic Group Theory
- Factoring / RSA Assumptions
- DLOG Assumption

Basic Number Theory

- $a \mid b$: a divides b ($b = a \cdot c$)
- Primes: an integer $p > 1$ that only has 2 divisors: 1 & p.
- Modular Arithmetic:

  $a \bmod N$: remainder of a when divided by N

  $a \cdot b \bmod N = (a \bmod N) \cdot (b \bmod N) \bmod N$.

  $a \equiv b \pmod N$: a and b are congruent modulo N

  How to compute $a^b \bmod N$? Time complexity? $O(\text{poly}(n))$

  a, b, N all $O(n)$ bits

$a \bmod N$

$a^2 \bmod N$

$a^4 \bmod N$

$\vdots$

$a^{2^n} \bmod N$

$$b = \sum_{i=0}^{n} b_i \cdot 2^i$$

$$a^b \equiv a^{\sum_{i=0}^{n} b_i \cdot 2^i} \equiv \prod_{i=0}^{n} (a^{2^i})^{b_i} \bmod N$$

Basic Number Theory

- gcd(a,b): greastest common divisor

How to compute gcd(a,b)? Time complexity? O(n)

a, b both O(n) bits

Euclidean Alg.

gcd(17, 12) = 1
17 mod 12 = 5
12 mod 5 = 2
5 mod 2 = 1
2 mod 1 = 0

gcd(18, 12) = 6
18 mod 12 = 6
12 mod 6 = 0

- gcd(a, N) = 1 : a & N are coprime

⇒ ∃b s.t. $a \cdot b \equiv 1 \pmod N$: a is invertible modulo N,
b is its inverse, denoted as a^{-1}.

How to compute b?

Extended Euclidean Alg.

a N
gcd(17, 12) = 1
17 mod 12 = 5
12 mod 5 = 2
5 mod 2 = 1
2 mod 1 = 0

5 = 17 − 12 × 1
2 = 12 − 5 × 2
1 = 5 − 2 × 2

gcd(a, N) = 1
⇓
$1 = a \cdot x + N \cdot y$
⇓ mod N
$1 \equiv a \cdot x$

Basic Number Theory

$\mathbb{Z}_N^* := \{a \mid a \in [1, N-1],\ \gcd(a, N) = 1\}$

Euler's phi (totient) function $\phi(N) := |\mathbb{Z}_N^*|$

Thm Let $N = \prod_{i=1}^{k} p_i^{e_i}$. p_i: distinct primes. $e_i \geq 1$.

Then $\phi(N) = \prod_{i=1}^{k} p_i^{e_i - 1}(p_i - 1)$.

Example: N is prime. $\phi(N) = N - 1$.

$N = p \cdot q$. $\phi(N) = (p-1) \cdot (q-1)$.

Euler's Theorem $\forall a, N$ where $\gcd(a, N) = 1$, $a^{\phi(N)} \equiv 1 \pmod N$.

Corollary If $d \equiv e^{-1} \bmod \phi(N)$, then $(a^d)^e \equiv a \bmod N$.

$\uparrow$

$d \cdot e \equiv 1 \bmod \phi(N)$

$\Downarrow$

$d \cdot e = \phi(N) \cdot Q + 1$

$\parallel$

$a^{d \cdot e} = a^{\phi(N) \cdot Q + 1} \bmod N$

$= 1^Q \cdot a \bmod N$

$= a \bmod N$

Factoring Assumption

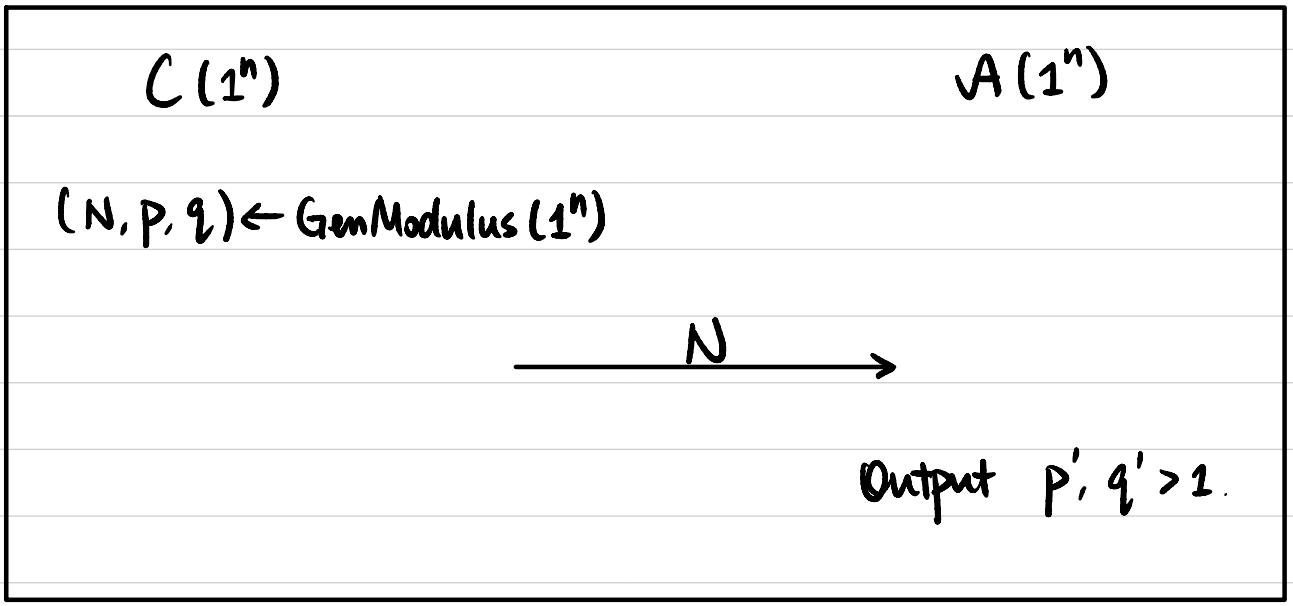
randomly sample → primality test

GenModulus (1^n): PPT algorithm, generates (N, p, q) ← How to generate?

p, q: n-bit primes, $p \neq q$. $N = p \cdot q$

Def Factoring is hard relative to GenModulus if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[p' \cdot q' = N] \leq \varepsilon(n)$.



Factoring ⇒ OWF (GenModulus)

RSA Assumption

GenModulus(1^n): generates (N, p, q). p, q: n-bit primes, $p \neq q$. $N = p \cdot q$

GenRSA(1^n):

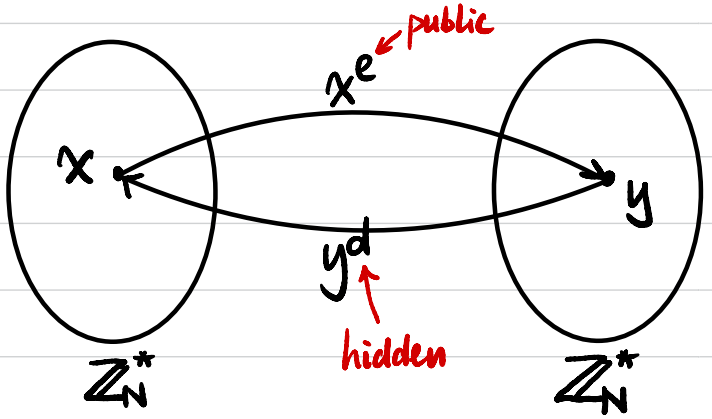
$(N, p, q) \leftarrow \text{GenModulus}(1^n)$

$\phi(N) := (p-1)(q-1)$ ← prime

Choose $e > 1$ s.t. $\gcd(e, \phi(N)) = 1$

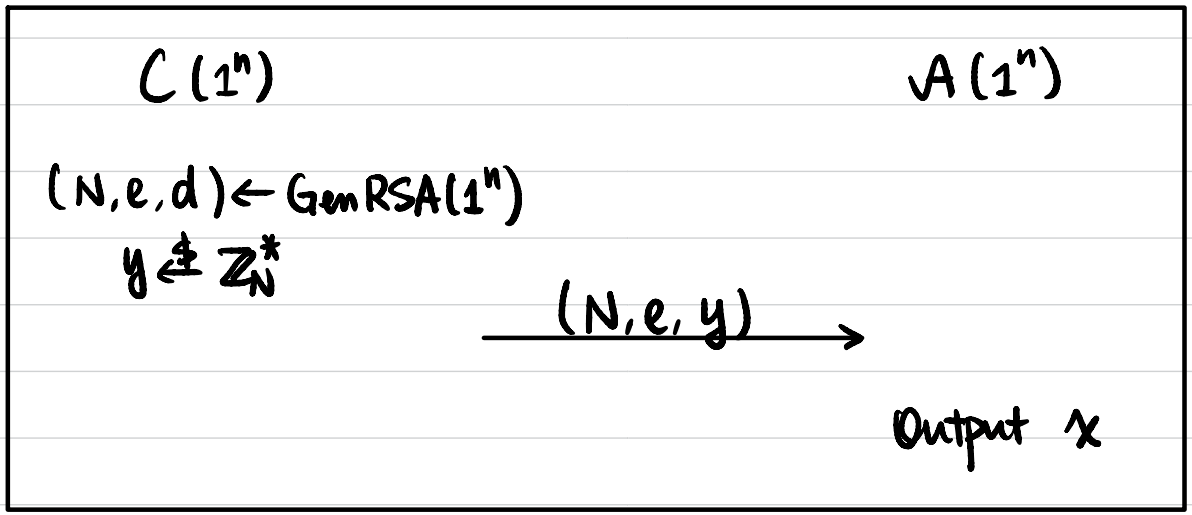
Compute $d := e^{-1} \bmod \phi(N)$

Output (N, e, d)



Def The RSA problem is hard relative to GenRSA if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[x^e = y \bmod N] \leq \varepsilon(n)$.



RSA $\Rightarrow$ Factoring

Basic Group Theory

Def A group is a set G along with a binary operation $\circ$ with properties:

① Closure: $\forall g, h \in G,\ g \circ h \in G$

② Existence of an identity: $\exists e \in G$ s.t. $\forall g \in G,\ e \circ g = g \circ e = g$.

③ Existence of inverse: $\forall g \in G,\ \exists h \in G$ s.t. $g \circ h = h \circ g = e$

Inverse of g denoted as g^{-1}.

④ Associativity: $\forall g_1, g_2, g_3 \in G,\ (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

Exercises: Is this a group?

- $(\mathbb{Z}, +)$ Yes
- $(\mathbb{Z}, \cdot)$ No
- $(G = \{0, 1, \cdots, N-1\},\ + \bmod N)$ Yes
- $(\mathbb{Z}_N^*,\ \cdot \bmod N)$ Yes

Basic Group Theory

Def We say a group is **abelian** if it satisfies:

⑤ Commutativity: $\forall g, h \in G,\ g \circ h = h \circ g$

For a finite group, we use $|G|$ to denote its **order** (# of elements)

$(H, \circ)$ is a **subgroup** of $(G, \circ)$ if $(H, \circ)$ is a group and $H \subseteq G$.

Group Exponentiation

For a group $(G, \cdot)$, $g^m := \underbrace{g \cdot g \cdots g}_{m \text{ times}}$ $\quad g^0 := 1$ (identity) $\quad g^{-m} := (g^{-1})^m$

$g^{m_1} \cdot g^{m_2} = g^{m_1+m_2}$ $\quad (g^{m_1})^{m_2} = g^{m_1 \cdot m_2}$ $\quad g^m \cdot h^m = (g \cdot h)^m$ $\quad g^{-m} = (g^m)^{-1}$

Thm Let G be a finite group of order m, then $\forall g \in G,\ g^m = 1$.

$\forall$ integer x, $g^x = g^{x \bmod m}$.

Proof $g_1 \cdot g_2 \cdots g_m = (g \cdot g_1)(g \cdot g_2) \cdots (g \cdot g_m)$

Basic Group Theory

Def Let G be a finite group of order m.

$\forall g \in G$, $\langle g \rangle := \{g^0, g^1, \cdots, g^{m-1}\}$ is a subgroup of G.

$|\langle g \rangle|$ is the **order** of g.

G is a **cyclic group** if $\exists g \in G$ s.t. $\langle g \rangle = G$. g is a **generator** of G.

Examples: ① If G is a group of prime order, then G is cyclic.

$\forall g \in G$, $g \neq 1$. g is a generator of G.

p, q primes $p = 2q + 1$.

Let $g \in \mathbb{Z}_p^*$ be an element of order q.

$H = \langle g \rangle$ is a group of prime order.

② $\mathbb{Z}_p^*$ for a prime p is a group of order $p-1$.

$p = 7$, $\langle 3 \rangle = \{1, 3, 2, 6, 4, 5\}$

Discrete-Log Assumption

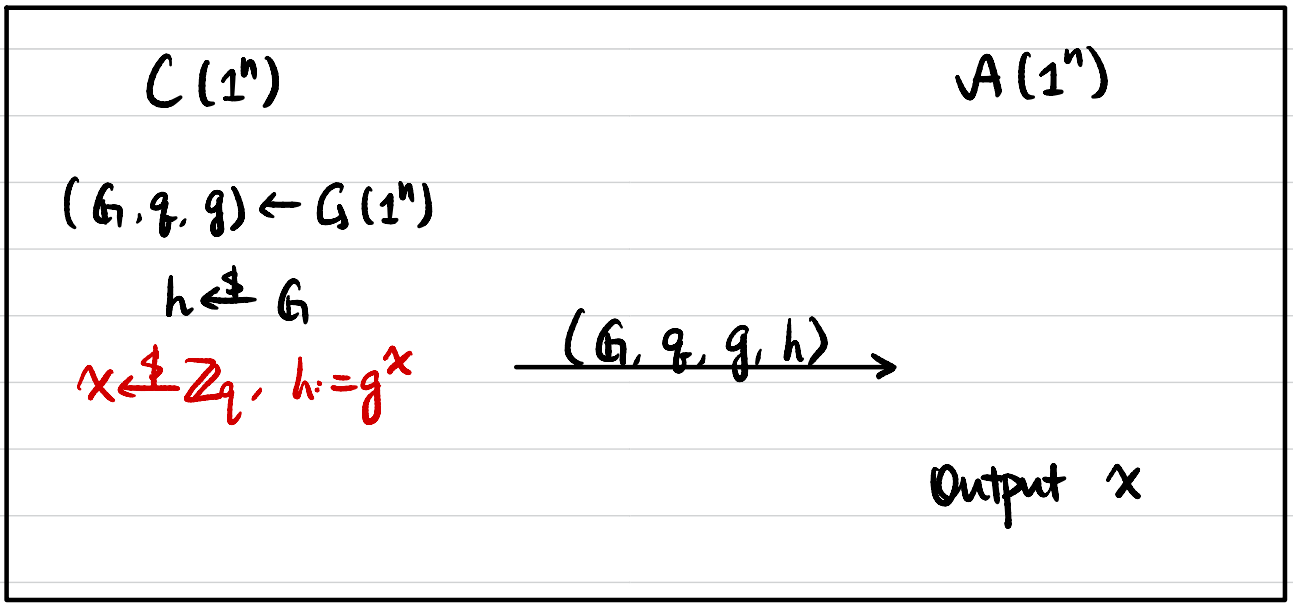
$G(1^n)$: PPT algorithm, generates (G, q, g)

description of a cyclic group G of order q with generator g.

(q: n-bit integer)

Def Discrete-Log (DLOG) is hard relative to G if

$\forall$ PPT A, $\exists$ negligible function $\varepsilon(\cdot)$ s.t. $\Pr[g^x = h] \leq \varepsilon(n)$.



DLOG $\Rightarrow$ CRHF

CRHF from DLOG Assumption

- $\text{Gen}(1^n)$: (q prime)

  $(G, q, g) \leftarrow \mathcal{G}(1^n)$

  $h \xleftarrow{\$} G$

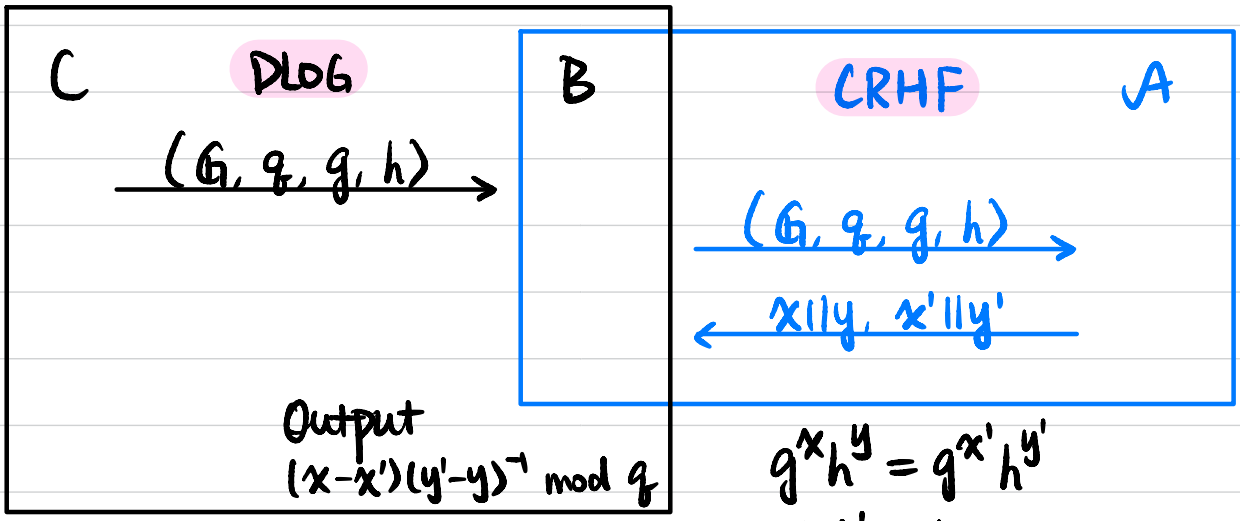
  Output $s = (G, q, g, h)$

- $H^s(x \| y) := g^x h^y$

Thm If DLOG is hard relative to $\mathcal{G}$, then this is a CRHF.

Proof Assume not, then $\exists$ PPT $\mathcal{A}$ that breaks collision resistence of H.

We construct PPT $\mathcal{B}$ to break DLOG.



$$g^x h^y = g^{x'} h^{y'}$$

$$\Rightarrow g^{x-x'} = h^{y'-y}$$

$$\Rightarrow g^{(x-x')(y'-y)^{-1}} = h$$