























Wikipedia – Fourier transform

Natural image

Natural image



Fourier decomposition Frequency coefficients (amplitude)



What does it mean to be at pixel x,y? What does it mean to be more or less bright in the Fourier decomposition image?

Basis reconstruction



Full image



First 9 basis fns

First 1 basis fn



First 16 basis fns



First 4 basis fns



First 400 basis fns

Danny Alexander

Properties of Fourier Transforms

• Linearity F[ax(t)+by(t)] = a F[x(t)]+b F[y(t)]

• Fourier transform of a real signal is symmetric about the origin

• The energy of the signal is the same as the energy of its Fourier transform

The Convolution Theorem

• The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathbf{F}[g * h] = \mathbf{F}[g]\mathbf{F}[h]$$

• **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

$$g * h = F^{-1}[F[g]F[h]]$$

Filtering in spatial domain

10-120-210-1





Slide: Hoiem

Now we can edit frequencies!



Low and High Pass filtering



Removing frequency bands



Brayer

High pass filtering + orientation









Application: Hybrid Images

When we see an image from far away, we are effectively subsampling it!





A. Oliva, A. Torralba, P.G. Schyns, SIGGRAPH 2006

Hybrid Image in FFT



Why does the Gaussian filter give a nice smooth image, but the square filter give edgy artifacts?



Why do we have those lines in the image?

 Sharp edges in the image need _all_ frequencies to represent them.



Box filter / sinc filter duality

- What is the spatial representation of the hard cutoff (box) in the frequency domain?
- http://madebyevan.com/dft/



Evan Wallace demo

- Made for CS123
- 1D example
- Forbes 30 under 30

- Figma (collaborative design tools)

http://madebyevan.com/dft/





with Dylan Field

Box filter / sinc filter duality

- What is the spatial representation of the hard cutoff (box) in the frequency domain?
- http://madebyevan.com/dft/



Spatial Domain \iff Frequency Domain Frequency Domain \iff Spatial Domain



Gaussian filter duality

- Fourier transform of one Gaussian... ...is another Gaussian (with inverse variance).
- Why is this useful?
 - Smooth degradation in frequency components

Frequency domain

magnitude

- No sharp cut-off
- No negative values
- Never zero (infinite extent)



Is convolution invertible?

• If convolution is just multiplication in the Fourier domain, isn't deconvolution just division?

- Sometimes, it clearly is invertible (e.g. a convolution with an identity filter)
- In one case, it clearly isn't invertible (e.g. convolution with an all zero filter)
- What about for common filters like a Gaussian?

Convolution



10

8

6

2

0

-2

-4





FFT











Deconvolution?



But under more realistic conditions



200

400

600

1000

800

Random noise, .000001 magnitude

-10

-20

-30

-40

But under more realistic conditions



200

400

600

1000

800

Hays

1000

-10

-20

-30

-40

Random noise, .0001 magnitude

200

400

600

800

But under more realistic conditions



Hays

But you can't invert multiplication by 0!

• A Gaussian is only zero at infinity...



Deconvolution is hard.

• Active research area.

• Even if you know the filter (non-blind deconvolution), it is still hard and requires strong *regularization* to counteract noise.

• If you don't know the filter (blind deconvolution), then it is harder still.

Fourier, Joseph (1768-1830)



French mathematician who discovered that any periodic motion can be written as a superposition of sinusoidal and cosinusoidal vibrations. He developed a mathematical theory of heat sin *Théorie Analytique de la Chaleur (Analytic Theory of Heat)*, (1822), discussing it in terms of differential equations.

Fourier was a friend and advisor of Napoleon. Fourier believed that his health would be improved by wrapping himself up in blankets, and in this state he tripped down the stairs in his house and killed himself. The paper of Galois which he had taken home to read shortly before his death was never recovered.

SEE ALSO: Galois

Additional biographies: MacTutor (St. Andrews), Bonn

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How would math have changed if the onesie had been invented?!?! : (

A few questions...

If we have infinite frequencies, why does the image end?

- Sampling theory. Frequencies higher than Nyquist frequency end up falling on an existing sample.
 - i.e., they are 'aliases' for existing samples!
- Nyquist frequency is half the sampling frequency.
- (Nyquist frequency is not Nyquist-Shannon rate, which is sampling required to reconstruct aliasfree signal. Both are derived from same theory.)

A few questions

Why is frequency decomposition centered in middle, and duplicated and rotated?

– From Euler:

• $\cos(x) + i\sin(x) = e^{ix}$

•
$$\cos(\omega t) = \frac{1}{2} \left(e^{-i\omega t} + e^{+i\omega t} \right)$$

 Coefficients for negative frequencies (i.e., 'backwards traveling' waves)

FFT of a real signal is conjugate symmetric

A few questions

How is the Fourier decomposition computed?

Intuitively, by correlating the signal with a set of waves of increasing signal!

Notes in hidden slides.

Plus: http://research.stowersinstitute.org/efg/Report/FourierAnalysis.pdf

Applications of Fourier analysis

- Fast filtering with large kernels
- Fourier Optics
 - Fraunhofer diffraction is Fourier transform of slit in the 'far field'.





Circular aperture = Airy disc diffraction pattern

Light spectrometry for astronomy

Thinking in Frequency - Compression

How is it that a 4MP image can be compressed to a few hundred KB without a noticeable change?

Lossy Image Compression (JPEG)

8x8 blocks



The first coefficient B(0,0) is the DC component, the average intensity

The top-left coeffs represent low frequencies, the bottom right represent high frequencies



Block-based Discrete Cosine Transform (DCT)

Slides: Efros

Image compression using DCT

• Compute DCT filter responses in each 8x8 block

Filter responses

- -30.19 -61.2027.2456.13 - 20.10 - 2.39-415.380.464.47 - 21.86 - 60.76 10.2513.15-7.09 - 8.544.88 $G = \begin{bmatrix} 4.47 & -21.80 & -00.70 & 10.25 & 10.15 \\ -46.83 & 7.37 & 77.13 & -24.56 & -28.91 \\ -48.53 & 12.07 & 34.10 & -14.76 & -10.24 \\ 12.12 & -6.55 & -13.20 & -3.95 & -1.88 \\ -7.73 & 2.91 & 2.38 & -5.94 & -2.38 \\ -1.03 & 0.18 & 0.42 & -2.42 & -0.88 \\ -0.17 & 0.14 & -1.07 & -4.19 & -1.17 \end{bmatrix}$ 9.935.42-5.656.301.831.951.75-2.793.140.944.301.85-3.024.12-0.66-0.100.501.68
- Quantize to integer (div. by magic number; round)
 - More coarsely for high frequencies (which also tend to have smaller values)
 - Many quantized high frequency values will be zero

Quantization divisers (element-wise)

Q =	16	11	10	16	24	40	51	61	
	12	12	14	19	26	58	60	55	
	14	13	16	24	40	57	69	56	
	14	17	22	29	51	87	80	62	
	18	22	37	56	68	109	103	77	
	24	35	55	64	81	104	113	92	
	49	64	78	87	103	121	120	101	
	72	92	95	98	112	100	103	99	

Quantized values

JPEG Encoding

• Entropy coding (Huffman-variant)

Quantized values

Linearize *B* like this.



Helps compression:

- We throw away the high frequencies ('0').
 - The zig zag pattern increases in frequency space, so long runs of zeros.







T (Cb=0.5,Cr=0.5)





Cb (Y=0.5,Cr=0.5)

Cr (Y=0.5,Cb=05)



Most JPEG images & videos subsample chroma



PSP Comp 3 2x2 Chroma subsampling 285K Original 1,261K lossless 968K PNG

JPEG Compression Summary

- 1. Convert image to YCrCb
- 2. Subsample color by factor of 2
 - People have bad resolution for color
- 3. Split into blocks (8x8, typically), subtract 128
- 4. For each block
 - a. Compute DCT coefficients
 - b. Coarsely quantize
 - Many high frequency components will become zero
 - c. Encode (with run length encoding and then Huffman coding for leftovers)