Naive Reconstruction:

We define the non-beard subspace as $\mathbf{V}$, where we make the projection of the our original image as such:

$$\begin{aligned} \mathbf{V} &\in \mathbb{R}^{d\times n} \\ \mathbf{x}^{\star} &= \mathbf{V}\hat{c} \end{aligned}$$

with

$$\hat{\mathbf{c}} = \arg\min_{\mathbf{c}} \|\mathbf{x} - \mathbf{V}\mathbf{c}\|_2^2 = \left(\mathbf{V}^T\mathbf{V}\right)^{-1}\mathbf{V}^T\mathbf{x}$$

Robust Reconstruction:

If we treat the beards as "outliers" of the face, we can try to isolate the pixels of great difference from the rest of the image. In place of the basic least squares, we must instead choose the Iteratively Reweighted Least Squares algroithm, with objective function

$$\begin{aligned} \hat{\mathbf{c}} &= \arg\min_{\mathbf{c}} \sum_{i=1}^{d} \rho\left(x_i - \tilde{\mathbf{v}}_i^T\mathbf{c}, \sigma\right) \\ \rho\left(x, \sigma\right) &= \frac{x^2}{x^2 + \sigma^2} \end{aligned}$$

This function does not have a closed-form solution, and as the name suggests, we must solve this numerically. ρ is know as the Geman-Mclure function. We can then determine how we create a diagonal bias matrix $\mathbf{W} \in \mathbb{R}^{d\times d}$, so that we can define our projection

$$\begin{aligned} \mathbf{c}^{(k)} &= \arg\min_{c} \|\mathbf{W}\left(\mathbf{x} - \mathbf{V}\mathbf{c}\right)\|_2^2 \\ &= \left(\mathbf{V^T W^T W V}\right)^{-1}\mathbf{V^T W^T W x} \end{aligned}$$

Then the solution for each index in $\mathbf{W}$ is

$$w_{ii} = \frac{1}{2e_i}\frac{\partial\rho\left(e_i, \sigma\right)}{\partial e_i} = \frac{\sigma^2}{\left(e_i^2 + \sigma^2\right)^2}$$

, with residual $\mathbf{e}$ at the k^{th} iteration of the algorithm as the difference between the original image, and σ defined in the paper

$$\begin{aligned} \mathbf{e} &= \mathbf{x} - \mathbf{V}\mathbf{c}^{(k-1)} \\ \sigma &\doteq 1.4826 * \text{median}\left(\{x_i - x_i^* : i = 1, \ldots, d\}\right) \end{aligned}$$

Then the solution to the unshaved image is the converged estimate:

$$\mathbf{x}^{\star} = \lim_{k\to\infty} \mathbf{V}\hat{\mathbf{c}}$$