[3] The Vector Space

Linear Combinations

An expression

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

is a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

The scalars $\alpha_1, \ldots, \alpha_n$ are the *coefficients* of the linear combination.

Example: One linear combination of $[2, 3.5]$ and $[4, 10]$ is

$$-5\,[2, 3.5] + 2\,[4, 10]$$

which is equal to $[-5 \cdot 2, -5 \cdot 3.5] + [2 \cdot 4, 2 \cdot 10]$

Another linear combination of the same vectors is

$$0\,[2, 3.5] + 0\,[4, 10]$$

which is equal to the zero vector $[0, 0]$.

Definition: A linear combination is *trivial* if the coefficients are all zero.

Linear Combinations: JunkCo

The JunkCo factory makes five products:

using various resources.

	metal	concrete	plastic	water	electricity
garden gnome	0	1.3	0.2	0.8	0.4
hula hoop	0	0	1.5	0.4	0.3
slinky	0.25	0	0	0.2	0.7
silly putty	0	0	0.3	0.7	0.5
salad shooter	0.15	0	0.5	0.4	0.8

For each product, a vector specifying how much of each resource is used per unit of product.

For making one gnome:

$\mathbf{v}_1 = \{\texttt{metal:0, concrete:1.3, plastic:0.2, water:.8, electricity:0.4}\}$

Linear Combinations: JunkCo

For making one gnome:
$\mathbf{v}_1 = \{$`metal:0, concrete:1.3, plastic:0.2, water:0.8, electricity:0.4`$\}$
For making one hula hoop:
$\mathbf{v}_2 = \{$`metal:0, concrete:0, plastic:1.5, water:0.4, electricity:0.3`$\}$
For making one slinky:
$\mathbf{v}_3 = \{$`metal:0.25, concrete:0, plastic:0, water:0.2, electricity:0.7`$\}$
For making one silly putty:
$\mathbf{v}_4 = \{$`metal:0, concrete:0, plastic:0.3, water:0.7, electricity:0.5`$\}$
For making one salad shooter:
$\mathbf{v}_5 = \{$`metal:1.5, concrete:0, plastic:0.5, water:0.4, electricity:0.8`$\}$

Suppose the factory chooses to make α_1 gnomes, α_2 hula hoops, α_3 slinkies, α_4 silly putties, and α_5 salad shooters.

Total resource utilization is $\mathbf{b} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 + \alpha_5 \mathbf{v}_5$

Linear Combinations: JunkCo: Industrial espionage

Total resource utilization is $\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 + \alpha_4\,\mathbf{v}_4 + \alpha_5\,\mathbf{v}_5$

Suppose I am spying on JunkCo.

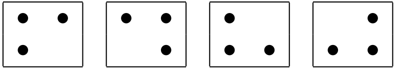
I find out how much metal, concrete, plastic, water, and electricity are consumed by the factory. That is, I know the vector $\mathbf{b}$. Can I use this knowledge to figure out how many gnomes they are making?

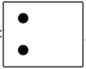
Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

- *input:* a vector $\mathbf{b}$ and a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* a list $[\alpha_1, \ldots, \alpha_n]$ of coefficients such that $\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$ or a report that none exists.

Question: Is the solution unique?

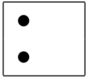
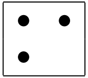
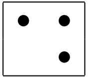
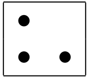
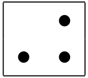
Lights Out

Button vectors for 2×2 *Lights Out:* 

For a given initial state vector $\mathbf{s} =$ , Which subset of button vectors sum to $\mathbf{s}$?

Reformulate in terms of linear combinations.

Write

 $= \alpha_1$  $+ \alpha_2$  $+ \alpha_3$  $+ \alpha_4$ 

What values for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ make this equation true?

Solution: $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0$

Solve an instance of *Lights Out* $\Rightarrow$ Which set of button vectors sum to $\mathbf{s}$?

$\Rightarrow$ Find subset of $GF(2)$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ whose sum equals $\mathbf{s}$ $\Rightarrow$ Express $\mathbf{s}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$

Lights Out

We can solve the puzzle if we have an algorithm for

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

Span

Definition: The set of all linear combinations of some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called the *span* of these vectors

Written Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Span: Attacking the authentication scheme

If Eve knows the password satisfies

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= \beta_m \end{aligned}$$

Then she can calculate right response to any challenge in Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$:

Proof: Suppose $\mathbf{a} = \alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m$. Then

$$\begin{aligned} \mathbf{a} \cdot \mathbf{x} &= (\alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m) \cdot \mathbf{x} \\ &= \alpha_1 \mathbf{a}_1 \cdot \mathbf{x} + \cdots + \alpha_m \mathbf{a}_m \cdot \mathbf{x} && \text{by distributivity} \\ &= \alpha_1 (\mathbf{a}_1 \cdot \mathbf{x}) + \cdots + \alpha_m (\mathbf{a}_m \cdot \mathbf{x}) && \text{by homogeneity} \\ &= \alpha_1 \beta_1 + \cdots + \alpha_m \beta_m \end{aligned}$$

Question: Any others? Answer will come later.

Span: $GF(2)$ vectors

Quiz: How many vectors are in Span $\{[1,1],[0,1]\}$ over the field $GF(2)$?

Answer: The linear combinations are

$$\begin{aligned}
0\,[1,1] + 0\,[0,1] &= [0,0] \\
0\,[1,1] + 1\,[0,1] &= [0,1] \\
1\,[1,1] + 0\,[0,1] &= [1,1] \\
1\,[1,1] + 1\,[0,1] &= [1,0]
\end{aligned}$$

Thus there are four vectors in the span.

Span: $GF(2)$ vectors

Question: How many vectors in Span $\{[1,1]\}$ over $GF(2)$?

Answer: The linear combinations are

$$0\,[1,1] = [0,0]$$
$$1\,[1,1] = [1,1]$$

Thus there are two vectors in the span.

Question: How many vectors in Span $\{\}$?

Answer: Only one: the zero vector

Question: How many vectors in Span $\{[2,3]\}$ over $\mathbb{R}$?

Answer: An infinite number: $\{\alpha\,[2,3]\ :\ \alpha \in \mathbb{R}\}$
Forms the line through the origin and $(2,3)$.

Generators

Definition: Let $\mathcal{V}$ be a set of vectors. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors such that $\mathcal{V} = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then

- we say $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a *generating set* for $\mathcal{V}$;
- we refer to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as *generators* for $\mathcal{V}$.

Example: $\{[3, 0, 0], [0, 2, 0], [0, 0, 1]\}$ is a generating set for $\mathbb{R}^3$.

Proof: Must show two things:

1. Every linear combination is a vector in $\mathbb{R}^3$.
2. Every vector in $\mathbb{R}^3$ is a linear combination.

First statement is easy: every linear combination of 3-vectors over $\mathbb{R}$ is a 3-vector over $\mathbb{R}$, and $\mathbb{R}^3$ contains all 3-vectors over $\mathbb{R}$.

Proof of second statement: Let $[x, y, z]$ be any vector in $\mathbb{R}^3$. I must show it is a linear combination of my three vectors....

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

Generators

Claim: Another generating set for $\mathbb{R}^3$ is $\{[1,0,0],[1,1,0],[1,1,1]\}$

Another way to prove that every vector in $\mathbb{R}^3$ is in the span:

- We already know $\mathbb{R}^3 = \text{Span }\{[3,0,0],[0,2,0],[0,0,1]\}$,
- so just show $[3,0,0]$, $[0,2,0]$, and $[0,0,1]$ are in Span $\{[1,0,0],[1,1,0],[1,1,1]\}$

$$\begin{array}{rcrcrcr} [3,0,0] & = & 3[1,0,0] & & & & \\ [0,2,0] & = & -2\,[1,0,0] & + & 2\,[1,1,0] & & \\ [0,0,1] & = & & & -1\,[1,1,0] & + & 1\,[1,1,1] \end{array}$$

Why is that sufficient?

- We already know any vector in $\mathbb{R}^3$ can be written as a linear combination of the old vectors.
- We know each old vector can be written as a linear combination of the new vectors.
- We can convert *a linear combination of linear combination of new vectors* into *a linear combination of new vectors*.

Generators

We can convert *a linear combination of linear combination of new vectors* into *a linear combination of new vectors*.

- Write $[x, y, z]$ as a linear combination of the old vectors:

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

- Replace each old vector with an equivalent linear combination of the new vectors:

$$\begin{aligned}[x, y, z] = (x/3)\,\Big(3\,[1, 0, 0]\Big) \quad &+ \quad (y/2)\Big(-2\,[1, 0, 0] + 2\,[1, 1, 0]\Big) \\ &+ \quad z\,\Big(-1\,[1, 1, 0] + 1\,[1, 1, 1]\Big)\end{aligned}$$

- Multiply through, using distributivity and associativity:

$$[x, y, z] = x\,[1, 0, 0] - y\,[1, 0, 0] + y\,[1, 1, 0] - z\,[1, 1, 0] + z\,[1, 1, 1]$$

- Collect like terms, using distributivity:

$$[x, y, z] = (x - y)\,[1, 0, 0] + (y - z)\,[1, 1, 0] + z\,[1, 1, 1]$$

Generators

Question: How to write each of the old vectors $[3,0,0]$, $[0,2,0]$, and $[0,0,1]$ as a linear combination of new vectors $[2,0,1]$, $[1,0,2]$, $[2,2,2]$, and $[0,1,0]$?

Answer:

$$[3,0,0] = 2\,[2,0,1] - 1\,[1,0,2] + 0\,[2,2,2]$$
$$[0,2,0] = -\frac{2}{3}\,[2,0,1] - \frac{2}{3}\,[1,0,2] + 1\,[2,2,2]$$
$$[0,0,1] = -\frac{1}{3}\,[2,0,1] + \frac{2}{3}\,[1,0,2] + 0\,[2,2,2]$$

Quiz: Writing new generators in terms of old generators

Old: $[2, 0, 1]$ and $[-4, 3, -2]$
New: $[2, 0, 1]$ and $[0, 1, 0]$.
Quiz: Write each of the new generators as a linear combination of the old generators.

Standard generators

Writing $[x, y, z]$ as a linear combination of the vectors $[3, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 1]$ is simple.

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

Even simpler if instead we use $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$:

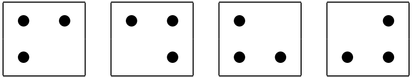
$$[x, y, z] = x\,[1, 0, 0] + y\,[0, 1, 0] + z\,[0, 0, 1]$$

These are called *standard generators* for $\mathbb{R}^3$.
Written $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

Standard generators

Question: Can 2×2 *Lights Out* be solved from every starting configuration?

Equivalent to asking whether the 2×2 button vectors are generators for $GF(2)^D$, where $D = \{(0,0), (0,1), (1,0), (1,1)\}$.



Yes! For proof, we show that each standard generator can be written as a linear combination of the button vectors: