Lecture 16
Recursion

Andrew, Janet, Emma & Jeff Like Cookies (1/2)
• They would each like to have one of these cookies:
• The first character who picks has four choices
• Three choices are left for the second character
• Two choices left for the third character
• The last character has to take what remains (poor Jeff!)

Andrew, Janet, Emma & Jeff Like Cookies (2/2)
• Thus we have 24 different ways the characters can choose cookies (4! = 4 x 3 x 2 x 1 = 24)
• What if we wanted to solve this problem for all the contestants on Survivor?
Factorial Function

- Model this problem mathematically:
  
  factorial \((n)\) calculates the total number of unique permutations, or the number of different ways to arrange/order \(n\) items.

- Small examples:
  
  \[
  1! = 1 \\
  2! = 2*1 = 2 \\
  3! = 3*2*1 = 6 \\
  4! = 4*3*2*1 = 24 \\
  5! = 5*4*3*2*1 = 120
  \]

- Iterative definition: \(n! = n * (n-1) * (n-2) * ... * 1\)

- Recursive definition: \(n! = n * (n-1)!\) for \(n \geq 1\) and \(0! = 1\)

Recursion (1/2)

- Models problems that are self-similar
  
  - decompose a whole task into smaller, similar sub-tasks
  - each subtask can be solved by applying the same technique

- Whole task solved by combining solutions to sub-tasks
  
  - special form of divide and conquer at every level

Recursion (2/2)

- Task is defined in terms of itself
  
  - in Java, recursion is modeled by method that calls itself, but each time with simpler case of the problem, hence the recursion will "bottom out" with a base case eventually
  - base case is a case simple enough to be solved directly, without recursion; otherwise infinite recursion and StackOverflowError
  - what is the base case of the factorial problem?
  - Java will bookkeep each invocation of the same method just as it does for nested methods that differ, so there is no confusion
  - usually you combine the results from the separate invocations
Factorial Function Recursively (1/2)

• Recursive factorial algorithm
  - the factorial function is not defined for negative numbers
    • the first conditional checks for this precondition
    • it is good practice to document and test preconditions
      (see code example)
  - number of times method is called is the depth of recursion (1 for 0!)
    • what is depth of (4)!?

```java
public class RecursiveMath {
    // instance variables, other code elided
    public int factorial (int num) {
        if (num < 0) {
            System.out.println("Input must be >= 0");
            return -1; // return -1 for invalid input
        }
        int result = 0;
        if (num == 0) {
            // base case: 0! = 1
            result = 1;
        } else {
            // general case
            result = num * this.factorial(num - 1);
        }
        return result;
    }
}
```

Factorial Function Recursively (2/2)

4! = factorial(4)
    = 4 * 3!
    = 4 * 3 * 2!
    = 4 * 3 * 2 * 1!
    = 4 * 3 * 2 * 1 * 0!
    = 24

TopHat Question

Given the following non-practical code:
```java
public class RecursiveMath {
    public int recursiveAddition(int n) {
        if (n <= 1) {
            return 1;
        } else {
            return this.recursiveAddition(n - 1);
        }
    }
}
```

What is the output of this.recursiveAddition(4)?

A. 1
B. 9
C. 10
D. StackOverflowError
TopHat Question

Given the following code:

```java
public class RecursiveMath {
    public int funkyFactorial(int n) {
        if (n == 0) {
            return 1;
        } else {
            return n * this.funkyFactorial(n-2);
        }
    }
}
```

What is the output of `this.funkyFactorial(5)`?

A. 1  
B. 5  
C. 15  
D. StackOverflowError

If you want to know more about recursion...

The following anecdote is told of William James. After a lecture on cosmology and the structure of the solar system, James was accosted by a little old lady. “Your theory that the sun is the center of the solar system, and the earth is a ball which rotates around it has a very convincing ring to it, Mr. James, but it’s wrong. I’ve got a better theory,” said the little old lady. “And what is that, madam?” inquired James politely. “That we live on a crust of earth which is on the back of a giant turtle.” Not wishing to demolish this absurd little theory by bringing to bear the masses of scientific evidence he had at his command, James decided to gently dissuade his opponent by making her see some of the inadequacies of her position. “If your theory is correct, madam,” he asked, “what does this turtle stand on?” “You’re a very clever man, Mr. James, and that’s a very good question,” replied the little old lady. “But I have an answer to it. And it’s this: The first turtle stands on the back of a second, far larger, turtle, who stands directly under him.” “But what does this second turtle stand on?” persisted James patiently. “It’s no use, Mr. James,” crowed triumphantly, “it’s turtles all the way down.”

— J. R. Ross, Constraints on Variables in Syntax 1967

Turtles in Recursion – from Wikipedia

The following anecdote is told of William James. After a lecture on cosmology and the structure of the solar system, James was accosted by a little old lady. “Your theory that the sun is the center of the solar system, and the earth is a ball which rotates around it has a very convincing ring to it, Mr. James, but it’s wrong. I’ve got a better theory,” said the little old lady. “And what is that, madam?” inquired James politely. “That we live on a crust of earth which is on the back of a giant turtle.” Not wishing to demolish this absurd little theory by bringing to bear the masses of scientific evidence he had at his command, James decided to gently dissuade his opponent by making her see some of the inadequacies of her position. “If your theory is correct, madam,” he asked, “what does this turtle stand on?” “You’re a very clever man, Mr. James, and that’s a very good question,” replied the little old lady. “But I have an answer to it. And it’s this: The first turtle stands on the back of a second, far larger, turtle, who stands directly under him.” “But what does this second turtle stand on?” persisted James patiently. “It’s no use, Mr. James,” crowed triumphantly, “it’s turtles all the way down.”

— J. R. Ross, Constraints on Variables in Syntax 1967
Call Out the Turtles

- Benoit Mandelbrot developed Fractals, a mathematical branch whose principle characteristic is self-similarity at any scale, one of the characteristics of recursions. Fractals are common in nature (botany, lungs, blood vessels, kidneys...), cosmology, antennas, $Z_{n+1} = Z_n^2 + C$ in the complex plane ($i$, where $i = \sqrt{-1}$)...
  
  - check out:
    - https://www.youtube.com/watch?v=2JUAojvFpCo
    - https://www.youtube.com/watch?v=aSg2Db3jF_4
    - http://matek.hu/xaos/doku.php
    - https://www.youtube.com/watch?v=4LQvjSf6SSw

- Some simpler, non-fractal, but still self-similar shapes composed of smaller, simpler copies of some pattern are simple spirals, trees, and snowflakes

- We can draw these using Turtle graphics
  
  - iteratively: Start at a particular point, facing in a chosen direction (here up). Draw successively shorter lines, each line at a given angle to the previous one
  - recursively: Start at a particular point, in a given direction. Draw a line of passed-in length, turn the passed-in angle, decrement length and call spiral recursively

Designing Spiral Class (1/2)

- Spiral class defines single draw method
  - turtle functions as pen to draw spiral, so class needs reference to turtle instance
  
- Constructor’s parameters to control its properties:
  - position at which spiral starts is turtle’s position
  - length of spiral’s starting side
  - angle between successive line segments
  - amount to change length of spiral’s side at each step
  
  Note: this info is passed to each invocation of recursive method, so next method call depends on previous one

Designing Spiral Class (2/2)

```java
public class Spiral {
    private Turtle turtle;
    private double angle;
    private int lengthDecrement;
    // passing in parameters to set the properties of the spiral
    public Spiral(Turtle myTurtle, double myAngle, int myLengthDecrement) {
        this.turtle = myTurtle;
        this.angle = myAngle;
        this.lengthDecrement = 1; // default, handles bad parameters
        if (myLengthDecrement > 0) {
            this.lengthDecrement = myLengthDecrement;
        }
        // draw method defined soon...
    }
}
```
Drawing Spiral

- First Step: Move turtle forward to draw line and turn some degrees. What's next?
- Draw smaller line and turn! Then another, and another...

Sending Recursive Messages (1/2)

- draw method uses turtle to trace spiral
- How does draw method divide up work?
  - draw first side of spiral
  - then draw smaller spiral (this is where we implement recursion)

```java
public void draw(int sideLen){
    // general case: move sideLen, turn
    // and draw smaller spiral
    this.turtle.forward(sideLen);
    this.turtle.left(this.angle);
    this.draw(sideLen - this.lengthDecrement);
}
```

Sending Recursive Messages (2/2)

- What is the base case?
  - when spiral is too small to see, conditional statement stops method so no more recursive calls are made
  - since side length must approach zero to reach the base case of the recursion, the draw method is called recursively, passing in a smaller side length each time

```java
public void draw(int sideLen){
    // base case: spiral too small to see
    if (sideLen <= 3) {
        return;
    }
    // general case: move sideLen, turn
    // and draw smaller spiral
    this.turtle.forward(sideLen);
    this.turtle.left(this.angle);
    this.draw(sideLen - this.lengthDecrement);
}
```
Recursive Methods

- We are used to seeing a method call other methods, but now we see a method calling itself
- Method must handle successively smaller versions of original task

Method’s Variable(s)

- As with separate methods, each invocation of the method has its own copy of parameters and local variables, and shares access to instance variables
- Parameters let method invocations (i.e., successive recursive calls) “communicate” with, or pass info between, each other
- Java’s record of current place in code and current values of parameters and local variables is called the activation record
  - at base case, as many exist as depth of recursion
  - each activation of a method is stored on the activation stack (you’ll learn about stacks soon)

Spiral Activation

```java
draw(int sideLen)
```

Initial value of sideLen: 25
Length decrement: 11
TopHat Question

Given the following code for the Collatz conjecture:

```java
public class RecursiveMath {
    private int count;  // constructor elided. count gets 0, it records
                       // number of calls on count
    public int collatzCounter(int n) {
        this.count += 1;  // base case
        if (n == 1) {
            return 1;
        }
        else {
            if (n % 2 == 0) {  // if n is even
                return collatzCounter(n / 2);
            }
            else {  // if n is odd
                return collatzCounter(3 * n + 1);
            }
        }
    }
}
```

What is the value of `count` after calling `collatzCounter(5)`?

A. 4  
B. 5  
C. 6  
D. StackOverflowError

The Collatz conjecture is a conjecture in mathematics named after Lothar Collatz. It concerns a sequence defined as follows: start with any positive integer n. Then each term is obtained from the previous term as follows: if the previous term is even, the next term is one half the previous term. Otherwise, the next term is 3 times the previous term plus 1. The conjecture is that no matter what value of n, the sequence will always reach 1. (From Wikipedia)

Towers of Hanoi (1/4)

- Game invented by French mathematician Édouard Lucas in 1883
- Goal: move tower of n disks, each of a different size (in order, with smallest at top), from left-most peg to right-most peg
- Rule 1: no disk can be placed on top of a smaller disk to win
- Rule 2: only one disk can be moved at a time

One Disk Solution

Move to target peg

Base Case
Two Disk Solution

Three Disk Solution

Pseudocode for Towers of Hanoi (1/2)

- Try solving for 5 non-recursively...
- One disk:
  - move disk to final pole
- Two disks:
  - use one disk solution to move top disk to intermediate pole
  - use one disk solution to move bottom disk to final pole
  - use one disk solution to move top disk to final pole
- Three disks:
  - use two disk solution to move top disks to intermediate pole
  - use one disk solution to move bottom disk to final pole
  - use two disk solution to move top disks to final pole
Pseudocode for Towers of Hanoi (2/2)

- In general (for \( n \) disks)
  - use \( n-1 \) disk solution to move top disks to intermediate pole
  - use one disk solution to move bottom disk to final pole
  - use \( n-1 \) disk solution to move top disks to final pole

- Note: can have multiple recursive calls in a method

---

Lower level pseudocode

// \( n \) is number of disks, \( src \) is starting pole,
// \( dst \) is finishing pole
public void hanoi(int \( n \), Pole \( src \), Pole \( dst \), Pole \( other \))
{
  if (\( n == 1 \)) {
    this.move(\( src \), \( dst \));
  }
  else {
    this.hanoi(\( n - 1 \), \( src \), \( other \), \( dst \));
    this.move(\( src \), \( dst \));
    this.hanoi(\( n - 1 \), \( other \), \( dst \), \( src \));
  }
}

public void move(Pole \( src \), Pole \( dst \))
{
  // take the top disk on the pole \( src \) and make
  // it the top disk on the pole \( dst \)
}

---

Fibonacci Sequence (1/2)

- 1, 1, 2, 3, 5, 8, 13, 21...
- Each number is calculated by adding the two previous numbers
  - \( F_n = F_{n-1} + F_{n-2} \)

See a fun application of Fibonacci sequence [here](#).
Fibonacci Sequence (2/2)

• What is the base case?
  o there are two: n=0 and n=1
• Otherwise, add two previous values of sequence together
  o this is also two recursive calls!

TopHat Question

Given the following code:

```java
public int fib(int n){
  //error check
  if (n < 0) {
    return -1;
  }
  //base case
  if (n == 0 || n == 1) {
    return 1;
  }
  //general case: add previous two values
  return fib(n-1) + fib(n-2);
}
```

What number would be returned if you excluded n == 1 from the base case and called fib(2)?

A. 5  B. 3  C. 2  D. 1

Loops vs. Recursion (1/2)

• Spiral uses simple form of recursion
  o each sub-task only calls on one other sub-task
  o this form can be used for the same computational tasks as iteration
  o loops (iteration) and simple recursion are computationally equivalent in the sense of producing the same result, if suitably coded (not necessarily the same performance, though — looping is more efficient)
Loops vs. Recursion (2/2)

• Iteration is often more efficient in Java because recursion takes more method calls (each activation record takes up some of the computer’s memory).
• Recursion is more concise and more elegant for tasks that are “naturally” self-similar (Towers of Hanoi is very difficult to solve iteratively!)

```java
public void drawIteratively(int sideLen){
    while(sideLen > 3){
        this.turtle.forward(sideLen);
        this.turtle.left(_angle);
        sideLen -= this.lengthDecrement;
    }
}
```

Indirect Recursion

• Two or more methods act recursively instead of just one
• For example, methodA calls methodB which calls methodA again
• Methods may be implemented in same or different classes
• Can be implemented with more than two methods too

Recursive Binary Tree (1/2)

• The tree is composed of a trunk that splits into two smaller branches that sprout in opposite directions at the same angle.
• Each branch then splits as the trunk did until sub-branch is deemed too small to be seen. Then it is drawn as a leaf.
• The user can specify the length of a tree’s main trunk, the angle at which branches sprout, and the amount by which to decrement each branch.
Recursive Binary Tree (2/2)

- Compare each left branch to its corresponding right branch
  - right branch is simply rotated copy
- Branches are themselves smaller trees!
  - branches are themselves smaller trees!
- Our tree is self-similar and can be programmed recursively!
  - base case is leaf

Designing the Tree Class

- Tree has properties that user can set:
  - start position (myTurtle’s built-in position)
  - angle between branches (myBranchAngle)
  - amount to change branch length (myTrunkDecrement)
- Tree class will define a single draw method
  - like Spiral, also uses a Turtle to draw

```java
class Tree {
    private Turtle turtle;
    private double branchAngle;
    private int trunkDecrement;

    public Tree(Turtle myTurtle, double myBranchAngle, int myTrunkDecrement) {
        this.turtle = myTurtle;
        if (myTrunkDecrement > 0) {
            this.trunkDecrement = myTrunkDecrement;
        } else {
            this.trunkDecrement = 1;
        }
        if (myBranchAngle > 0) {
            this.branchAngle = myBranchAngle;
        } else {
            this.branchAngle = 45;
        }
        // draw method coming up...
    }

    private void draw(int trunkLen) {
        if (trunkLen <= 0) {
            this.addLeaf();
        } else {
            this.turtle.forward(trunkLen);
            this.turtle.left(this.branchAngle);
            this.draw(trunkLen - this.trunkDecrement);
            this.turtle.right(2 * this.branchAngle);
            this.draw(trunkLen - this.trunkDecrement);
            this.turtle.left(this.branchAngle);
            this.turtle.back(trunkLen);
        }
    }
}
```

Tree’s draw Method

- Base case: if branch size too small, add a leaf
- General case:
  - move Turtle forward
  - orient Turtle left
  - recursively draw left branch
  - orient Turtle right
  - recursively draw right branch
  - reset Turtle to starting orientation
  - back up to prepare for next branch
Overall Program View

/* Class that creates a tree and utilizes its recursive methods in order to draw it. */
public class BuildTree {
    private Tree myTree;

    public BuildTree() {
        Turtle turtle = new Turtle();
        double branchAngle = 30;
        int trunkDecrement = 1;
        int trunkLen = 6;
        // Remember that draw() in Tree class took in a trunkLen
        this.myTree = new Tree(turtle, branchAngle, trunkDecrement);
        this.createTree(trunkLen);
    }

    // Method that is going to call draw recursively to draw our tree!
    public void createTree(int currTrunkLen) {
        this.myTree.draw(currTrunkLen);
    }
}

Recursive Snowflake

• Invented by Swedish mathematician, Helge von Koch, in 1904; also known as Koch Island
• Snowflake is created by taking an equilateral triangle and partitioning each side into three equal parts. Each side’s middle part is then replaced by another equilateral triangle (with no base) whose sides are one third as long as the original.
  • The user can specify the length of the initial equilateral triangle’s side
  • “Mathematical monster”: infinite length with a bounded area
Snowflake's draw Method

- Can draw equilateral triangle iteratively
- `drawSnowFlake` draws the snowflake by drawing smaller, rotated triangles on each side of the triangle (compare to iterative `drawTriangle`)
- `for` loop iterates 3 times
- Each time, calls the `drawSide` helper method (defined in the next slide) and reorients `turtle` to be ready for the next side

```java
public void drawTriangle(int sideLen) {
    for (int i = 0; i < 3; i++) {
        this.turtle.forward(sideLen);
        this.turtle.right(120.0);
    }
}
```

```java
public void drawSnowFlake(int sideLen) {
    for (int i = 0; i < 3; i++) {
        this.drawSide(sideLen);
        this.turtle.right(120.0);
    }
}
```

Snowflake's drawSide method

- `drawSide` draws single side of a recursive snowflake by drawing four recursive sides
- Base case: simply draw a straight side
- `MIN_SIDE` is a constant we set indicating the smallest desired side length
- General case: draw complete recursive side, then reorient for next recursive side

```java
private void drawSide(int sideLen) {
    if (sideLen < MIN_SIDE) {
        this.turtle.forward(sideLen);
    } else {
        this.drawSide(Math.round(sideLen / 3));
        this.turtle.left(60.0);
        this.drawSide(Math.round(sideLen / 3));
        this.turtle.right(120.0);
        this.drawSide(Math.round(sideLen / 3));
        this.turtle.left(60.0);
        this.drawSide(Math.round(sideLen / 3));
    }
}
```

Hand Simulation

1) Call `draw(90)`, which calls `drawSide(90)`, which calls `drawSide(30)`, which calls `drawSide(10)`. Base case reached because 10 < MIN_SIDE.
2) `drawSide(10)` returns to `drawSide(30)`, which tells this `turtle` to turn left 60 degrees and then calls `drawSide(10)` for a second time.
3) `drawSide(10)` returns to `drawSide(30)`, which tells this `turtle` to turn right 120 degrees and then calls `drawSide(10)` for a third time.
4) `drawSide(10)` returns to `drawSide(90)`, which tells this `turtle` to turn left 60 degrees and then calls `drawSide(90)` for a fourth time.

After this call, `drawSide(90)` returns to `drawSide(90)`, which resides in `turtle` and calls `drawSide(90)` again.
Again: Koch Snowflake Progression

Summary
• Recursion models problems that are self-similar, decomposing a task into smaller, similar sub-tasks.
• Whole task solved by combining solutions to sub-tasks (divide and conquer)
• Since every task related to recursion is defined in terms of itself, method will continue calling itself until it reaches its base case, which is simple enough to be solved directly.

Announcements
• DoodleJump Information
  o on-time handin: 11/3
  o late handin: 11/5
  o snake mock check-in now up!
• Lab 7 Arrays this week:
  o pre-lab video and pre-lab quiz
What is labor in the tech industry?

- Labor refers to everyone who develops technical products, from software engineers to assembly workers.
- Vast disparity in working conditions and pay.

Labor in the tech industry

- Often we think of labor as encompassing:
  - High-paid workers including software engineers, product managers, designers
  - Gig work and creatives ("non-technical" counterparts)
  - Support staff at tech companies: custodial staff, culinary staff
  - Factory labor
  - Clockwork platform labor
  - Mining labor
Abusive Labor Practices

Content warning: suicide

• "Foxconn Suicides": series of suicides in 2010 linked to low pay and meticulous, grueling workplace conditions at Foxconn electronics manufacturing plant in Shenzhen, China
  • Suicide nets (photographed on right)
  • Foxconn a major manufacturer of Apple products
  • In 2019 China Labor Watch accused Apple of withholding bonus payments, rolling back safety training, and employing more temporary workers than the law allows

Gig-economy and subcontracting

• Tech companies avoid recognizing workers as employees
• Child labor in DRC cobalt mines
  • Tech companies filed to have the suit dismissed, arguing they are not responsible
  • Since they don’t own cobalt mines and use intermediary suppliers, specific sites cannot be identified as sources
• Proposition 22
  • 2020 CA initiative permitting companies to treat ride-share and delivery drivers as independent contractors, not employees
  • 5 major gig companies spent over $200 million to pass Prop 22!
    • why: employees are entitled to fair wages and labor protections, contractors are not
  • Uber, Lyft etc. save $$$
• Content Moderation
  • Facebook outsources content moderation to India, Philippines
Unionizing Bessemer

- Amazon warehouse workers in Alabama voted on unionizing - lost
- Coercion and threat of consequences interfered with employees' freedom of choice
- "Information sessions" and text messages, signage posted around the facility
- Odds often stacked in favor of the employers
- ~1800 vs. 738
- Many wanted to protect $15/hr minimum wage, health insurance
- High rate of employee turnover

More reading that may be of interest!

- Google Office in Dublin — Archity Peerview
- Atlas of AI by Kate Crawford
- Apple and Google named in US lawsuit over Congolese child cobalt mining — the Guardian
- Tech is Splitting the U.S. Work Force in Two — New York Times
- Life and death in Apple's forbidden city — FoxConn
- Voices of the Valley by Ben Tarnoff
- Prop 22 was a failure for California's App-based workers — NELP
- It's a No: Amazon Warehouse Workers Vote Against Unionizing in Historic Election — NPR
- Why Amazon Workers sided with the company over a Union — NYTimes