Homework #5

Out : October 7, 2:30 pm 1997
Due : October 14, 12:30 pm 1997

1. The DigiComp computer can add a pair of digits (and a carry, if needed) in one clock tick. Thus, to add two ten-digit numbers, DigiComp requires ten ticks. To add a ten-digit number and a six-digit number, DigiComp takes six or seven ticks (depending on the final carry). For each of the given addition problems, tell us how many clock ticks DigiComp needs to solve the problem. Your answer need not be exact; it's okay to be off slightly.

   (a)
   ```
    12345678
   +    1234
   ----------------
   ```

   (b)
   ```
    123456789012345678
   +   123456789012345
   --------------------
   ```

   (c)
   ```
    12345678901234567890123456789012345678901234567890123456789012345678901234567890
   +12345678901234567890123456789012345678901234567890123456789012345678901234567890
   --------------------
   ```

   (d)
   ```
    123456789
    234567891
   + 345678912
   ------------
   ```

   (e)
   ```
    12345678901234567890
    23456789012345678901
    34567890123456789012
    45678901234567890123
    56789012345678901234
    67890123456789012345
    78901234567890123456
    89012345678901234567
    90123456789012345678
   +
   ----------------------
   ```

2. The DigitComp computer can multiply a pair of digits (and add in a carry, if needed) in one clock tick. Thus to multiply a twenty-digit number by a one-digit number, DigiComp requires twenty ticks. Multiplying a twenty-digit number by a two-digit number, say 37, involves (i) multiplying the twenty-digit number by 7, obtaining a twenty- or twenty-one-digit number, (ii) multiplying the twenty-digit number by 3, obtaining a twenty- or twenty-one-digit number, and shifting it to the left by one place, and (iii) adding the results of (i) and (ii). Part (i) takes twenty ticks, part (ii) takes twenty ticks, and part (iii) takes about twenty-one ticks. Thus the total number of ticks required is about sixty-one.

For each of the following multiplication problems, tell us how many clock ticks DigiComp needs to solve the problem. Show your work. (Your answer need not be exact; there is no problem with being off by a few.) **Don't be afraid to use short-cuts; i.e. if you think you see a pattern, use it.**

(a)
```
 12345678
x     123
---------
```

(b)
```
 12345
x 1234
-------
```

(c)
```
 12345678901234567890123456789012345678901
x                                 1234567890
--------------------------------------------
```

3. **This is the most important problem on this homework. It requires some care.** Suppose DigiComp needs to multiply two k-digit numbers. Which formula below most accurately expresses the number of ticks needed? Explain your answer by referring to the multiplication process. **Note: The formula you choose need not be exactly correct, it need only be the best estimate. The percentage error for the correct formula is smaller for larger values of k, while the percentage error for incorrect formulae tend to be larger for larger values of k. You can check your answer by seeing how accurately the formula you choose predicts the number of ticks required for multiplying, say, two ten-digit numbers. As in the previous problem, don't be afraid to use short-cuts: look for patterns.**

   - $k + 2$
   - $4k$
   - $2k\sqrt{k}$
   - $2k^2$
   - $k^3/8$
   - 2^k
   - 10^{k-2}

4. For each of the following formulae, imagine a computer for which that formula gave the number of ticks required to multiply two k-digit numbers. For which values of k would DigiComp be faster, and for which values of k would the imaginary computer be faster? Your answer need not be exact. **Hint: you may find it helpful to graph the functions.**

   (a) $10k + 4$

   (b) $k^2\sqrt{\sqrt{k}}$

   (c) $2^k/100$

   (d) $2^{\sqrt{k}}$

5. The base-10 logarithm of a number is roughly the number of digits comprising that number. Use this fact to estimate the base-10 logarithms given below. (Don't use a calculator.)

   (a) $\log_{10} 12345678901234567890$

   (b) $\log_{10} 12345678901234567890123456789012345678901234567890123456789012345678901234567890$

   (c) $\log_{10} 9999999999999$

   (d) $\log_{10} 1111111111111$

6. For each of the following, give a value of x for which the base-10 logarithm is approximately the number given. (There are many right answers.)

   (a) $\log_{10} x \approx 5$

   (b) $\log_{10} x \approx 7$

   (c) $\log_{10} x \approx 30$

7. Suppose DigiComp needs to multiply the numbers a and b. Which of the following formulae most accurately estimates the number of ticks required to perform the multiplication? Explain your answer, and compare it to the exact answer for $a = 1234567890, b = 1234567890$.

   - a/b
   - $4a^2b^2$
   - $2 \log_{10} ab$
   - $2(\log_{10} a)(\log_{10} b)$
   - $10^a 10^b$

8. Another imaginary computer requires approximately k^3 clock ticks to multiply two k-digit numbers. Which of the following formulae must accurately estimates the number of ticks this computer requires to multiply the numbers a and b? Explain your answer, and compare it to the exact answer for $a = 1234567890, b = 1234567890$.

   - a^3b^3
   - $(ab)^3$
   - $(10^a 10^b)^3$
   - $(\log_{10} a)^3(\log_{10} b)^3$
   - $\log_{10}(a^3b^3)$

9. The rules of exponentiation are

$$a^{b+c} = a^b a^c$$

   and

$$(a^b)^c) = a^{bc}$$

   Use these rules to simplify each of the following formulae.

   (a) $\frac{x^{15}}{x^7/(xy)^8}$

   (b) $(x^{20})(y^5y^{10} - y^{15})$

   (c) $y^{15} - \frac{y^4y^6}{y^3y^8}$