# Greedy Bidding Strategies for Keyword Auctions

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## ABSTRACT

How should players bid in keyword auctions such as those used by Google, Yahoo! and MSN? We consider greedy bidding strategies for a repeated auction on a single keyword, where in each round, each player chooses some optimal bid for the next round, assuming that the other players merely repeat their previous bid. We study the revenue, convergence and robustness properties of such strategies. Most interesting among these is a strategy we call the *balanced bidding strategy* (BB): it is known that BB has a unique fixed point with payments identical to those of the VCG mechanism. We show that if all players use the BB strategy and update each round, BB converges when the number of slots is at most 2, but does not always converge for 3 or more slots. On the other hand, we present a simple variant which is guaranteed to converge to the same fixed point for any number of slots. In a model in which only one randomly chosen player updates each round according to the BB strategy, we prove that convergence occurs with probability 1. We complement our theoretical results with empirical studies.

## 1. INTRODUCTION

Online search engine advertising is an appealing approach to highly targeted advertising, and is estimated to be *the* major source of revenue for modern web search engines such as Google and Yahoo! The basic setup is the following: When an individual does a query in a search engine, he gets back a page of results that contains the links the search engine has deemed relevant to the search, together with a small number of *sponsored links*, i.e., paid advertisements. The beauty of this from the advertiser's perspective is that they can precisely target their ads based on the search words used. For example, if a travel agent "buys" the search term "Tahiti" when a user searches on the word "Tahiti", a link to the web page for that travel agent offering, say, cheap fares to Tahiti, might appear as one of the sponsored links on the search results page. If the user actually clicks on this link, he will be transferred to the aforementioned web page. For each such click, in which the advertiser "receives" a potential customer, the advertiser pays the search engine.

The process of determining which ads get assigned to which keywords (search terms) and how much each advertiser pays is resolved via *keyword auctions*. Advertisers choose which keywords they want to bid on and participate in auctions for those keywords. For each keyword of interest, the advertiser submits a bid stating the maximum amount they are willing to pay for a click. When a user submits a query on that keyword, an instantaneous auction is run to determine which of the advertisers currently bidding on that keyword is allocated an advertising slot.

In the absence of budget constraints, there is one and only one *truthful* auction that can in principle be used: the Vickrey-Clarke-Groves or VCG mechanism [12, 3, 7]. This mechanism has the property that it is in the best interest of the participating advertisers to bid their true valuation of a click. Despite this appealing property of the VCG mechanism, for a number of reasons, no search engine uses the VCG mechanism. Rather, the most widely used auction mechanism is the non-truthful *Generalized Second Price* or GSP auction (described in Section 2).

The fact that the GSP mechanism is not truthful means that the participating advertisers are forced to undertake the complicated task of choosing a bidding strategy. Asdemir [1] and Edelman et al.[4] observe that instability and bidding wars can result from the use of the GSP mechanism. To make matters worse, on a typical search page, there is room for multiple sponsored links. The positioning of these sponsored links affects the chances that a sponsored link will be clicked on and thus these advertising slots have varying desirability from the perspective of advertisers. This makes the advertisers' utilities a discontinuous function of their bids. Overall, the resulting bidding is sufficiently complex that many advertisers hire consultants or intermediaries to do their bidding for them, often at significant cost.

In a typical day, an advertiser will choose one or more search terms to target and compete in a potentially very large number of keyword auctions for those terms. In between auctions for a particular keyword, the bidders have the opportunity to update their bids. This is typically done automatically via software robots [9].

In this paper, we undertake a systematic exploration of a very natural class of greedy bidding strategies a software robot might use in a repeated keyword auction for a partic-


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ular search term(s). The main theme is the following: If a particular advertiser $A$ knew how the other bidders were going to bid in the next round, $A$ would bid so as to maximize his utility. The greedy bidding strategies we study assume that the recent past is the best prediction for the future: $A$ assumes that the other bidders will bid in the next round exactly what they bid in the current round. Given this assumption, $A$ chooses his bid for the next round to maximize his utility relative to this postulated set of bids by the other bidders.

There is still quite a lot of flexibility within this definition. Suppose that, given the presumed bids by the other bidders, the optimal advertising slot for $A$ to target is slot $s$ at price $p_s$, and that the price of the (higher click-through rate) slot $s-1$ is $p_{s-1}$. The GSP mechanism allows a range of bid values that will result in the same outcome from $A$'s perspective (any bid between $p_s$ and $p_{s-1}$). We focus the bulk of our study on one particular choice in this range: *Balanced bidding* (BB), in which the advertiser chooses his next bid $b$ so as to be indifferent between successfully winning the targeted slot $s$ at price $p_s$, or winning a more desirable slot at price $b$. BB is a particularly interesting strategy, since a system in which all players bid according to BB has a unique fixed point in which players are bidding according to a Nash equilibrium of GSP and the payments to the search engine are identical to those of the VCG mechanism (see Theorem 5).

Our main results about BB are the following:

- For two slots, BB always converges to its unique fixed point (Theorem 6 part 1).
- For three or more slots, BB need not ever converge, assuming that in each round, *all* players simultaneously update their bids according to the BB greedy strategy (Theorem 6 part 2).
- In an asynchronous model, where exactly one randomly chosen player updates his bid each round according to the BB strategy, BB bidding always converges eventually (Theorem 6 part 4). This is not true if the bidders are not chosen randomly.
- We present a simple variation on BB with the same unique fixed point and prove that it converges to its fixed point even in the synchronous model in which all players update their bids according to BB at each step (Theorem 10).

An important perspective on these results is the following: The Nash equilibria of GSP are fairly straightforward to characterize and have been understood for some time [5, 11, 10]. What was not known was if there was a natural bidding strategy that would lead to these Nash equilibria. Here we show how players can get to the most natural of the GSP equilibria using a greedy strategy.

Given that the revenue of BB converges to that of VCG, one might ask: if bidders do not run BB how much revenue do the search engines obtain compared to the revenue they would get using VCG. In Section 5, we explore this question, assuming that bidders eventually end up in some Nash equilibrium of GSP (which of course is not necessarily the case). We present an empirical comparison of the VCG revenue with that of the minimum revenue GSP equilibrium and the maximum revenue GSP equilibrium. We complement these experiments with some theoretical bounds (Theorem 16).

We also compare the VCG revenue to that that would be obtained if the bidders used other natural greedy strategies: *competitor busting* (CB), in which the advertiser chooses the highest bid value consistent with the target slot, and *altruistic bidding* (AB), in which the advertiser chooses the lowest bid value consistent with the target slot. It is easy to see that except for degenerate circumstances, neither CB nor AB bidding has a fixed point. Thus, we empirically evaluate their average revenue over a long sequence of keyword auctions.

## 2. MODEL AND DEFINITIONS

**Definition** 1. *A* keyword auction *is defined by:*

- *A set of $k$ slots with click-through rates (CTRs) $\theta_1 > \ldots > \theta_k$, where $\theta_i$ is the probability that the user will click on the advertisement in slot $i$.*
- *A set of $n$ players (advertisers) participating in the auction, each one having a private valuation $v_i$ for a click, $v_1 > \ldots > v_n$.*
- *Based on knowledge of the auction mechanism and their own private valuations, each player submits a bid to the auction. We denote by $b_i$ the bid submitted by player $i$.*
- *The auction mechanism:*
  - *computes an allocation $\pi$ of the slots to $k$ different players: $\pi_s$ is the identity of the player that is allocated slot $s$.*
  - *charges a price $p_s$ to the player $\pi_s$ for each click on his advertisement.*
- *If player $i$ is allocated slot $s$ at price $p_s$, player $i$'s expected utility is $\theta_s(v_i - p_s)$.*

The *generalized second price* mechanism is the most common auction mechanism in use.

**Definition** 2. *The* generalized second price *mechanism (GSP) for keyword auctions uses the following allocation and payment rules:*

- *Players are allocated slots in decreasing order of bids.*[1]
- *For each slot $s$, the payment $p_s$ of player $\pi_s$ is $b_{\pi_{s+1}}$.*

*Players who do not win a slot make no payment and gain no utility.*

The GSP mechanism is not truthful, but has a continuum of Nash equilibria that are well understood [5, 11, 10]. One of these equilibria results in player payments identical to those that would be made if the mechanism being employed was VCG. We call this Nash equilibrium of GSP the *VCG*

[1]Technically, there is also a "relevance" or per-advertiser click-through rate $r_i$ associated with each advertiser $i$, and bidders are actually ranked (assigned slots) in decreasing order of ranking score, where the ranking score of bidder $i$ is $b_i r_i$. In this paper we assume $r_i = 1$ for all $i$. All results in the paper extend easily to the case where the $r_i$ values are arbitrary.

*equilibrium* of GSP. GSP also has Nash equilibria in which the revenue to the search engine is either higher or lower than that in the VCG equilibrium (see, for example, Varian [11] or Theorem 16 in this paper). It is not clear whether any of these equilibria are actually reached in real keyword auctions. It is also not clear what bidding strategy players might employ to reach a particular equilibrium. One of the goals of this paper is to gain some insight into these issues.

We consider a *repeated keyword auction*, with a fixed set of $n$ players and $k$ slots. The participants in the repeated auction have the opportunity to update their bids in between successive rounds. How should players bid? Naturally, a player's main objective is to maximize his own expected utility over multiple rounds of the auction. However, without any real insight into the bidding strategies followed by the rest of the players, it is difficult for one player to make predictions about the future bids of other players and hence choose an optimal bidding strategy. Thus, a very natural approach is to assume that the immediate past is the best predictor of the future. This leads to a natural greedy-like bidding scheme where a player assumes that all the other bids will remain fixed in the next round. Under this assumption, the rational choice for a player $j$ is to bid so as to win a slot $s$ that maximizes his utility $u_j = \theta_s(v_j - p_s)$. This leads to the following definition.

**Definition** 3. ***Greedy Bidding Strategies***
*A* greedy bidding strategy *for a player $j$ is to choose a bid for the next round of a repeated keyword auction round so as to maximize his utility $u_j$, assuming the bids of all other players $b_{-j}$ in the next round will remain fixed to their values in the previous round.*

Given $b_{-j}$, denote by $p_{-j}(s)$ the payment player $j$ would make if he bids so as to win slot $s$. Let $s^*$ be the slot the greedy bidder $j$ will target. Then if player $j$ is greedy, he will bid $b' \in (p_{-j}(s^*-1), p_{-j}(s^*))$. As $b'$ is not fully specified, this defines a class of strategies that are distinguished by the choice of $b'$ within the allowed range.

Since the advertisers participating in a keyword auction are often business competitors, one of the most common secondary objectives besides gaining the optimal slot observed in practice is the desire to "push" the prices paid by other advertisers higher. This is naturally done by bidding in the high end of the range mentioned earlier. However, this has an inherent risk as a change in other players' bids could result in paying a higher price than expected. This naturally leads to the following bidding strategy.

**Definition** 4. ***Balanced Bidding***
*The* Balanced *greedy strategy* BB *is the strategy for a player $j$ that, given $b_{-j}$*

- *next targets the slot $s_j^*$ which maximizes his utility (greedy bidding choice), that is,*

$$s_j^* = \text{argmax}_s\{\theta_s(v_j - p_{-j}(s))\}.$$

- *chooses its bid $b'$ for the next round so as to satisfy the following equation:*

$$\theta_{s_j^*}(v_j - p_{-j}(s_j^*)) = \theta_{s_j^*-1}(v_j - b').$$

*If $s_j^*$ is the top slot, we (arbitrarily) choose $b' = (v_j + p_{-j}(1))/2$. We can thus deal with all slots uniformly by defining $\theta_0 = 2\theta_1$.*

The intuition behind the bid selection is that the player bids high enough to force the prices paid by his competitors to rise, but not so high that if one of his competitors were to just undercut him, he would mind getting a higher slot at a price just below his own bid of $b'$. The BB strategy is an appealing strategy for the following reason.

**Theorem** 5. *[5] If all the players are following the* BB *strategy in an auction with all distinct $\theta$s, then the system has a unique fixed point. In this fixed point, the revenue of the auctioneer (and payments of each player) is equal to that of the VCG equilibrium. The equilibrium bids $b_j^*$ of the players in the fixed point of* BB *satisfy the following equations:*

$$b_j^* = \begin{cases} v_j & \text{if } j \geq k+1 \text{ and} \\ \gamma_j b_{j+1}^* + (1-\gamma_j)v_j & \text{if } 2 \leq j \leq k. \end{cases} \quad (1)$$

*where $\gamma_j = \theta_j/\theta_{j-1}$.*

## 3. CONVERGENCE PROPERTIES OF THE BB STRATEGY

We study the convergence properties of the BB strategy in a repeated keyword auction under two models. We refer to our standard model, where all players simultaneously update their bids on each round as the *synchronous* model. In the *asynchronous* model, in each round, exactly one player updates her bid, while the other players merely repeat their previous bids. We consider both the case in which the player performing the update is arbitrary and the case where the player performing the update is chosen at random. This has been studied, for example in Zhang [14]. The asynchronous model is closer to realistic applications, although the synchronous model does apply in on-line settings where bids are updated in batches.

**Theorem** 6. *Consider a repeated keyword auction where all players are using the* BB *strategy starting with arbitrary initial bids. We have:*

1. *A 2-slot auction system always converges to its fixed point in both the synchronous and asynchronous models. The number of rounds until convergence in the synchronous model is $O(\log((v_2 - v_3)/v_3))$, where the constant depends on the click-through rates $\theta_1$ and $\theta_2$.*

2. *There exists a 3-slot auction system and set of initial bids which does not converge in the synchronous model.*

3. *There exists a 3-slot auction system, a set of initial bids, and an order in which the players update which does not converge in the asynchronous model.*

4. *In the asynchronous model where players bid in random order, no matter how many slots there are, the system always converges to its fixed point.*

*Proof of Part 1:*

**Lemma** 7. *At every round $t$ such that $t > t_1 = 2 + \log_{\gamma^*}((1-\gamma^*)(v_2 - v_3)/v_3)$, where $\gamma^* = \max\{\theta_1/\theta_0, \theta_2/\theta_1\}$, we have:*

$$\begin{aligned} b_1, b_2 &> v_3, \\ b_i &= v_i, \quad \forall i \geq 3. \end{aligned}$$

PROOF. Let $b$ denote the third highest bid. By definition, $b$ can never be more than $v_3$. Suppose for some round that $b$ is less than $v_3$. Take a player $i$ in $\{1,2,3\}$. In the next round, $i$ will bid her value or target some slot $j \in \{1,2\}$ and bid $b_i' = (1-\gamma_j)v_i + \gamma_j p_j \geq (1-\gamma^*)v_3 + \gamma^* b = b + (1-\gamma^*)(v_3 - b)$. In either case,

$$(v_3 - b_i') \leq \gamma^*(v_3 - b).$$

Initially $v_3 - b \leq v_3$.

It takes at at most $r = \log_{\gamma^*}((1-\gamma^*)(v_2 - v_3)/v_3)$ before $v_3 - b < (1-\gamma^*)(v_2 - v_3)$? At most $r = \log_{\gamma^*}((1-\gamma^*)(v_2 - v_3)/v_3)$. In round $r+1$, bidders $i \in \{1,2\}$ will each bid either $v_i > v_3$ or target a slot $j \in \{1,2\}$ and bid

$$\begin{aligned} b_i' &= (1-\gamma_j)v_i + \gamma_j p_j \geq (1-\gamma^*)v_2 + \gamma^* b \\ &= b + (1-\gamma^*)(v_2 - b) > b + (1-\gamma^*)(v_2 - v_3) \\ &> v_3, \end{aligned}$$

hence in either case their bids are both above $v_3$. In round $r+2$, player 3 will then bid $v_3$ while players 1 and 2 keep on bidding above $v_3$; the other players can bid at most their value, which is less than $v_3$, and this concludes the proof of the lemma. □

From that point on, the price of slot 2 is fixed at $p_2 = v_3$. Let $T_2 = b_2^* = (1 - \theta_2/\theta_1)v_2 + (\theta_2/\theta_1)p_2$ and $T_1 = (1 - \theta_2/\theta_1)v_1 + (\theta_2/\theta_1)p_2$. If the last bids of players 1 and 2 were $b_1$ and $b_2$ then, in the next round, their bids are:

$$b_1' = \begin{cases} T_1 & \text{if } b_2 > T_1, \\ (v_1 + b_2)/2 & \text{otherwise.} \end{cases} \tag{2}$$

$$b_2' = \begin{cases} T_2 & \text{if } b_1 > T_2, \\ (v_2 + b_1)/2 & \text{otherwise.} \end{cases}$$

Let $b_{\min} = \min(b_1, b_2)$ be the minimum of the two bids.

**Lemma** 8. *After at most $t_2$ rounds, we have $b_{\min} \geq T_2$, where*

$$t_2 \leq t_1 + 2\frac{\theta_1 - \theta_2}{\theta_2}.$$

PROOF. Assume that at time $t > t_1$, we have $b_{\min} < T_2$. Then it is easy to check that at the next round we have $b_{\min}' \geq (v_2 + b_{\min})/2$. This implies

$$b_{\min}' - b_{\min} \geq \frac{v_2 - b_{\min}}{2} \geq \frac{v_2 - T_2}{2} = \frac{\theta_2}{\theta_1}\frac{v_2 - p_2}{2} = \delta.$$

Since at time $t_1$ we have $b_{\min} \geq v_3$, we will reach $b_{\min}' \geq T_2$ after an additional number of rounds bounded by $(T_2 - v_3)/\delta$. □

Finally, since at round $t_2$ we have $b_{\min} \geq T_2$, at time $t_2 + 1$ we will have $b_2' = T_2 < T_1$, and therefore at time $t_2 + 2$ we will have $b_2'' = T_2 = b_2^*$ and $b_1'' = (v_1 + T_2)/2 = b_1^*$; we have reached equilibrium. This proves Part 1 of the Theorem in the synchronous model.

To prove convergence in the asynchronous model, we will follow along the lines of the previous proof. Indeed, by focusing on the rounds where specific players get to bid, lemma 7 still holds but this time we need $r$ bid alternations between players $\{1,2,3\}$ and two additional alternations of players $\{1,2\}$. Lemma 8 is easier to check as in this case only the top two players alter their bids, therefore after $t_2$ bid alternations we reach the same outcome. ■

*Proof of Part 2:* Unfortunately, an auction system where all the players follow the BB strategy is not guaranteed to converge when there are more than 2 slots. We will show this by the following counterexample. Let there be three slots with $\theta_1 = 1$, $\theta_2 = 2/3$, $\theta_3 = 1/3$ and four players with values 161, 160, 159 and 100. Let the initial bids of 130.5, 130, 129.5 and 100, respectively. Then the bidding evolves as below; in particular it is not convergent.

| | Round | | |
|---|---|---|---|
| Bidder Value | 1 | 2 | 3 |
| 161 | 130.5 | 145.5 | 130.5 |
| 160 | 130 | 145.25 | 130 |
| 159 | 129.5 | 144.75 | 129.5 |
| 100 | 100 | 100 | 100 |

There is one inactive bidder, with the lowest value, who will never be able to bid for a slot as the prices are all above his utility. It the first round, the remaining bidders all target the lowest slot. The bid from the highest-valued player is low enough so that for the next round, all three top bidders target the first slot. With these high bids, the price for any player for the top slot is too high when compared with the price of the last slot. Thus for the third round all three top players target the last slot, and the cycle continues.

In this example the click-through rates follow a simple geometric sequence, similar to those observed in practice [6]. Note that in this example the bids are well-behaved, in the sense that the bids are in the same order as the players' values. Hence such regularity is not sufficient for convergence. Finally, even though we used our convention of $\theta_0 = 2$, a similar example can be constructed where the players are cycling while targeting intermediate slots. This concludes the proof of part 2. ■

*Proof of Part 3:* Consider the following three slot example with $\theta_1 = 1$, $\theta_2 = 0.1$ and $\theta_3 = 0.09$ and the top three bidders alternating in sequence.

| | Round | | | | | | |
|---|---|---|---|---|---|---|---|
| Value | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 102 | 19.2 | 80.8 | 80.8 | 80.8 | 19.2 | 19.2 | 19.2 |
| 101 | 19.1 | 19.1 | 90.9 | 90.9 | 90.9 | 19.1 | 19.1 |
| 100 | 59.6 | 59.6 | 59.6 | 95.45 | 95.45 | 95.45 | 59.6 |
| 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

■

*Proof of Part 4:* We will argue that there exists an asynchronous bidding sequence for the players that converges to the fixed point. A proof sketch is given in Appendix 5.5. ■

## Notes

- The example above showing that for 3 slots, BB doesn't converge in the synchronous model is not anomalous. In Section 5, we show experimentally that in a significant fraction of instances, BB does not converge in the synchronous model.
- The bound we give on the time to convergence of BB in the random asynchronous model is extremely loose. We have run simulations to study the speed of convergence in this setting and have found experimentally that convergence is actually quite fast, certainly no more than a polynomial in $n$, the number of bidders.

# 4. CONVERGENCE PROPERTIES OF THE RESTRICTED-BB STRATEGY

We will now examine a variant of the BB strategy, called RBB, where the players can only aim for their current slot or a slot of lower click-through rate than the one they currently have. The RBB strategy is designed so that it has the same unique fixed point as BB. However, by restricting the degree to which a player can be greedy, we will be able to show that even in the synchronous model, RBB always converges to the VCG equilibrium.

**Definition** 9. *The* Restricted Balanced Bidding *(*RBB*) bidding strategy is the strategy where given $b_{-j}$ from the previous round, player $j$*

- *targets the slot $s_j^*$ which maximizes his utility* among the slots with no higher click-through rate than his current slot $s_j$*, that is,*

$$s_j^* = \text{argmax}\{\theta_s(v_j - p_{-j}(s)) \ : \ s \geq s_j\}.$$

- *chooses her bid $b'$ for the next round so as to satisfy the following equation: $\theta_{s_j^*}\big(v_j - p_{-j}(s_j^*)\big) = \theta_{s_j^*-1}\big(v_j - b'\big)$.*

*To make sure this is well-defined for the first slot, we define $\theta_0 = 2\theta_1$.*

**Theorem** 10.

1. *The system defined by a repeated keyword auction in which all the players are following the* RBB *strategy has a unique fixed point at which players are bidding according to the VCG equilibrium, i.e. the equilibrium bids $b_j^*$ are given by Equation (1).*

2. *In the synchronous model the* RBB *strategy always converges to its fixed point. The number of steps until convergence is $2^k$ times*

$$O\left(k + \frac{\log(1-\gamma^*)}{\log \gamma^*} + \log_{(1/\gamma^*)} \frac{v_1 - v_{k+1}}{\min_{1\leq i \leq k}(v_i - v_{i+1})}\right),$$

*where $\gamma^* = \max_i \theta_i/\theta_{i-1}$.*

Proof. The proof of Part 1 of the Theorem is very similar to that of Theorem 5 (see Edelman et al.[5]). To prove Part 2, we first bound the number of steps until the price of slot $k$ and the set of players who will be allocated slots have converged. As before, we define $\gamma_i = \theta_i/\theta_{i-1}$.

**Lemma** 11. *Player $p$ prefers to target slot $j$ rather than slot $j-1$ if and only if*

$$(1-\gamma_j)v_p + \gamma_j p_j < p_{j-1}.$$

The proof of this Lemma is simple algebra and is omitted.

**Lemma** 12. *At every round $t$ such that $t > t_1 = 2 + \log_{\gamma^*}((1-\gamma^*)(v_k - v_{k+1})/v_{k+1})$, where $\gamma^* = \max_{i>0}\theta_i/\theta_{i-1}$, we have:*

$$\begin{aligned} b_i &> v_{k+1} \quad \forall i \leq k, \\ b_i &= v_i \quad \forall i \geq k+1. \end{aligned}$$

Proof. Let $b$ denote the $(k+1)$st highest bid. By definition, $b$ can never be more than $v_{k+1}$. Suppose for some round that $b$ is less than $v_{k+1}$. Take any player $i$ in $\{1, 2, \ldots, k+1\}$. In the next round, $i$ will either bid her value or target some slot $j \in \{1, \ldots k\}$ and bid $b_i' = (1-\gamma_j)v_i + \gamma_j p_j \geq (1-\gamma^*)v_{k+1} + \gamma^* b \geq b + (1-\gamma^*)(v_{k+1} - b)$. In either case,

$$(v_{k+1} - b_i') \leq \gamma^*(v_{k+1} - b).$$

Initially $v_{k+1} - b \leq v_{k+1}$. How many rounds does it take before $v_{k+1} - b < (1-\gamma^*)(v_k - v_{k+1})$? At most $r \leq \log_{\gamma^*}((1-\gamma^*)(v_k - v_{k+1})/v_{k+1})$. In round $r+1$, bidders $i \in \{1, \ldots k\}$ will each bid either $v_i > v_{k+1}$ or bid at least

$$\begin{aligned} b_i' &= (1-\gamma_j)v_i + \gamma_j p_j \geq (1-\gamma^*)v_k + \gamma^* b \\ &\geq b + (1-\gamma^*)(v_k - b) > b + (1-\gamma^*)(v_k - v_{k+1}) \\ &> v_{k+1}, \end{aligned}$$

hence in either case their bids are above $v_{k+1}$. In round $r+2$, player $k+1$ will then bid $v_{k+1}$ while players $1, 2, \ldots, k$ continue to bid above $v_{k+1}$; the other players don't get a slot, so they bid their value, and this concludes the proof. □

We now need to prove that the allocation of the $k$ slots players to these $k$ players converges to a fixed point.

At any time, for any $i \in [1, k]$, consider the players allocated slots $[i+1, k]$. They are called *stable* if their bids and prices satisfy Equation (1), that is, the allocation is in order of decreasing values, and if $\pi(j)$ is the player currently allocated slot $j$, then the last bids of those players satisfied:

$$b_{\pi(j)} = \gamma_j b_{\pi(j+1)} + (1-\gamma_j)v_{\pi(j)},$$

for every $j \in [i+1, k]$.

If the players allocated all slots $[1, k]$ form a stable set, then we have reached the fixed point of the RBB strategy.

Assume that the current setting is not (yet) a fixed point of the RBB strategy. Let $A$ be the maximum stable set, with associated $i \geq 2$, and let $B$ be the set of players in slots $[1, i]$. Let $b_{\min}$ denote the minimum bid from players of $B$.

We define a partial order over sets of players. We say that $A' \succ A$ if either $A \subset A'$ and $A \neq A'$, or if the smallest $v_p$ which is in the symmetric difference of $A$ and $A'$ belongs to $A'$ [2]. This will be our measure of progress.

In the next round, observe that, following the RBB strategy, players in $A$ still bid in the same way as before. Let $b'_{\min}$ be the new minimum bid from players of $B$, and $p$ be the player of $B$ whose bid is $b'_{\min}$. There are three cases to consider.

1. $p$ bids below $p_i$. Let $j \in [i+1, k]$ be the slot which was targeted by $p$. By definition, $p$ prefers slot $j$ to slot $j-1$, and so, by Lemma 11, the bid of $p$ is less than $p_{j-1}$. By definition, the bid is $(1-\gamma_j)v_p + \gamma_j p_j > p_j$, thus it falls in the interval $(p_{j-1}, p_j)$ and $p$ will be allocated slot $j$. Recall that $\pi(j) \in A$ denotes the player who was in slot $j$ in the previous round. Since set $A$ is stable, by definition we have $p_{j-1} = (1-\gamma_j)v_{\pi(j)} + \gamma_j p_j$. Since this is greater than the bid of $p$, it follows that $v_{\pi(j)} > v_p$. Moreover, since $p$ prefers slot $j$ to slot $j+1$, by Lemma 11 again, we must have $(1-\gamma_{j+1})v_p + \gamma_{j+1}p_{j+1} > p_j = (1-\gamma_{j+1})v_{\pi(j+1)} + \gamma_{j+1}p_{j+1}$, and so $v_p > v_{\pi(j+1)}$. Thus $A' = \{p' \in A : v_{p'} < v_p\} \cup \{p\}$ is a stable set, and $A' \succ A$.

2. $p$ targeted slot $i$. Then $p$ is allocated slot $i$, $A' = A \cup \{p\}$ is a stable set, and $A' \succ A$.

[2] This corresponds to a lexicographic ordering of the sets.

3. $p$ targeted some slot $j \leq i-1$. Then $A$ is still a stable set, and $b_{\min} = p_{i-1}$ has increased: $b'_{\min} = (1-\gamma_j)v_p + \gamma_j p_j \geq b_{\min} + (1-\gamma^*)(v_p - b_{\min})$.

We will prove that Case 3 can only happen a bounded number $x$ of times, (where $x$ depends on the $\theta_j$'s and the $v_j$'s but not on the bids) before Case 1 or 2 must occur. Thus, the maximum stable set must change at least once every $x$ rounds, and when that happens, it is replaced by a set which is larger in the $\succ$ ordering. This implies that the system converges to a fixed point and that the number of rounds until convergence is bounded by $2^k(x+1)$, hence the Theorem.

First, a useful technical lemma.

**Lemma** 13. *Let $\varepsilon = (1/2)\theta_k(1-\gamma^*)\min_{q\neq q'}|v_q - v_{q'}|/\theta_1$. If $p_{i-1} > v_p - \varepsilon$ and $v_p > p_i$, then player $p$ prefers slot $i$ to any slot $j < i$.*

PROOF. From player $p$'s viewpoint, the utility of slot $i$ is $\theta_i(v_p - p_i)$, the utility of slot $j < i$ is $\theta_j(v_p - p_j) < \theta_j(v_p - p_{i-1})$, and the ratio is

$$\frac{\theta_j(v_p - p_{i-1})}{\theta_i(v_p - p_i)} \leq \varepsilon\frac{\theta_j}{\theta_i(v_p - p_i)} \leq \varepsilon\frac{\theta_1}{\theta_k(v_p - p_i)}.$$

Now,

$$\begin{aligned} v_p - p_i &= v_p - ((1-\gamma_{i+1})v_{\pi(i+1)} + \gamma_{i+1}p_{i+1}) \\ &= (1-\gamma_{i+1})(v_p - v_{\pi(i+1)}) + \gamma_{i+1}(v_p - p_{i+1}), \end{aligned}$$

which is at least $(1-\gamma^*)\min_{q\neq q'}|v_q - v_{q'}|$. Plugging this into the previous expression proves the Lemma. □

Now, let $x = \log_{1/\gamma^*}((v_1 - v_{k+1})/\varepsilon)$. Assume that Case 3 happens for $x$ consecutive rounds. Let $p_{\min}$ be the player in $B$ whose value is minimum and $v_{\min}$ be its value. Let $b^{(t)}_{\min}$ be the minimum bid of players in $B$ after $t$ rounds, $0 \leq t \leq x$. If $p \in B$ is the player defining the minimum bid in round $t+1$, we have:

$$b^{(t+1)}_{\min} \geq (1-\gamma^*)v_p + \gamma^* b^{(t)}_{\min} \geq (1-\gamma^*)v_{\min} + \gamma^* b^{(t)}_{\min}.$$

After $x$ rounds, we get $b^{(x)}_{\min} \geq v_{\min} - (\gamma^*)^x(v_{\min} - b^{(0)}_{\min})$, hence $b^{(x)}_{\min} \geq v_{\min} - (\gamma^*)^x(v_i - v_{k+1})$. Plugging in the value of $x$ yields $b^{(x)}_{\min} \geq v_{\min} - \varepsilon$. From Lemma 13, we know that $p_{\min}$ prefers slot $i$ to any slot $j < i$. In the next round $p_{\min}$ targets slot $i$ and has to be the minimum bidder from $B$, therefore we are now in Case 2. Thus there are at most $x$ occurrences of Case 3 between any two occurrences of Case 1 or Case 2, and the Theorem is proved. ■

# 5. EMPIRICAL EVALUATION

In the previous part of the paper, we showed how simple greedy bidding strategies can lead to the VCG equilibrium. Is this the "right" equilibrium for bidders to be shooting for? How desirable is this outcome for the bidders and for the search engine? In this section, we study these questions empirically.

We first compare the search engine revenue in equilibrium to the VCG revenue. We then compare these benchmarks with two alternative greedy bidding strategies, both from the perspective of search engine revenue and from the perspective of bidder utility.

## 5.1 Experimental Setup

In all experiments, we define the keyword auction as follows: we use three slots and four players. Following the study of Feng et al.[6], we choose the click-through rates as a geometrically decreasing sequence by $\theta_i = \delta^{i-1}$ for some value of $\delta$ between 0 and 1, and plot our results as a function of $\delta$. We take the average of 150 instances where for each instance, the values of the four players are each independently chosen from a normal distribution with mean 500 and deviation 200.

To analyze a bidding strategy, for each of the 150 instances, the simulated strategy is run for 75 rounds from starting bids of 1 for all players, except for AB (defined later), which is run for 150 rounds from starting bids equal to the minimum player value. All experiments are run in the synchronous model of bidding.

## 5.2 Revenue in GSP Equilibria

How much revenue does GSP bring to search engines, compared to the revenue generated by the VCG mechanism?

Figure 1 plots the average, over instances, of the ratio between the revenue in some Nash equilibrium of GSP and the VCG revenue. Since GSP has many different Nash equilibria, we consider three extreme points: the maximum revenue Nash equilibrium, the minimum revenue Nash equilibrium, and the maximum revenue Nash equilibrium when the bidders are *debt-averse*, i.e. never bid above their value. Kitts et al.[8] show evidence that bidders usually follow a debt-averse bidding strategy.

From these experiments, we conclude the following: *the equilibrium revenue of GSP is within* $\pm 40\%$ *of the VCG revenue, unless $\delta$ is close to 1 and bidders are willing to bid above their value.* This should be contrasted with Part 1 of Theorem 16, where we exhibit an example of a GSP equilibrium with revenue much smaller than the VCG revenue: our simulations indicate that such an example is an anomaly.

We also see the following interesting behavior: *If bidders are debt-averse, then in equilibrium GSP generates at most* 15% *more revenue than VCG.* Since being debt-averse seems a very likely behavior on the part of rational bidders, this indicates that choosing GSP over VCG may not have a huge impact on revenue. The influence of debt-averse behavior on the maximum revenue is rigorously justified in Part 3 of Theorem 16.

## 5.3 Balanced Bidding Strategies: BB and RBB

We next demonstrate the frequency of non-convergence of BB shown in Theorem 6 part 2. Figure 2 plots the probability that BB converges. (When $\delta > .95$, it takes much longer for the system to reach a fixed point or a cycle, and our simulations did not always enable us to reach that point, and so we plot two curves representing lower and upper bounds to the convergence probability.) This figure indicates that the non-convergent example in Theorem 6 is not pathological: *The non-convergence of* BB *is a practically observed phenomenon which occurs for a significant fraction of instances.* One contribution of this paper is to suggest that the bidders use RBB rather than BB, since it gets to the same fixed point and has the advantage of always converging. However, players might wonder about the effect on their utility. To study this, we take one special player $p$ and fix his value at 500, while the values of the other players are

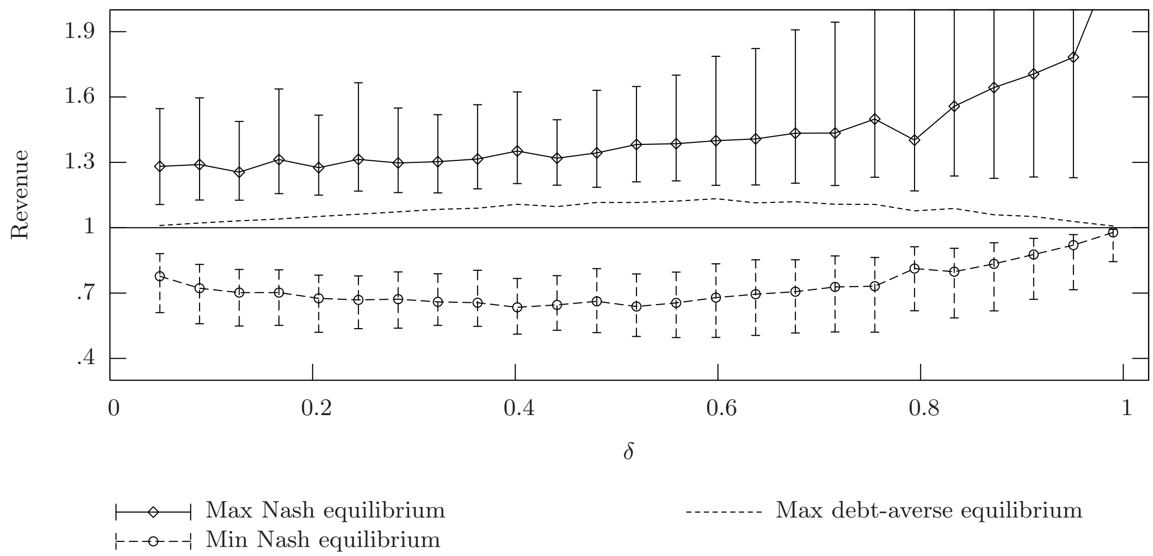


**Figure 1: Comparing revenue from GSP equilibria with the VCG revenue (VCG=1)**

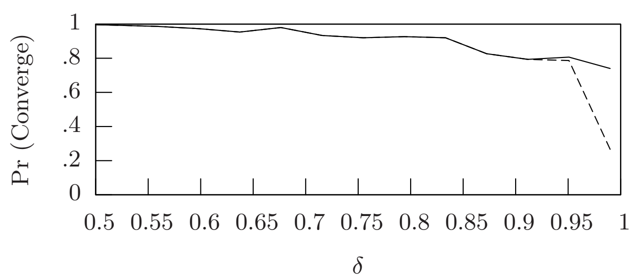


**Figure 2: Convergence of BB**

each independently chosen from a normal distribution with mean 500 and deviation 200. We then plot, as a function of $\delta$, the ratio between player $p$'s utility under RBB and his utility under BB.

We see in Figure 3, that when $\delta < .6$, player $p$'s utility in RBB is only about 90% of the utility he would get in BB. Interestingly, as $\delta$ approaches 1, this is reversed: when $\delta$ is close to 1, RBB yields better utility than BB. We expect this to be due to the frequent non-convergence of BB in this regime. It is claimed [6] that in practice the click-through rates are fit well by a geometric sequence with $\delta$ about .7, and in that range RBB and BB yield about the same revenue.

Note that our result is robust: we have observed that changing the value of the player under focus from 500 to 400 or to 600 (so as to make him an either relatively lower valued player or relatively higher valued player) does not change the graph significantly.

## 5.4 Other Greedy Bidding Strategies: CB and AB

So far we have focused on BB and RBB, but it is conceivable that a bidder could obtain a higher revenue by following some other strategy. In this section we examine two other greedy bidding strategies, one that is often considered and another that is its natural complement. In the following we let $\varepsilon_{\text{price}}$ be a suitably small bid increment, for example 1¢.

### 5.4.1 Competitor Busting

A popular bidding strategy used in practice is known as *competitor busting* [13]. This strategy is also referred to as anti-social or vindictive bidding [2, 15], and may be used by as many as 40% of the bidders on Yahoo! [15]. The idea is that a player bids as high as possible while retaining her desired slot, in order to make competitors pay as much as possible and exhaust their advertising resources.

**Definition** 14. *The* Competitor-Busting *greedy bidding strategy (*CB*) is the strategy for a player $j$ that, given $b_{-j}$*

- *next targets the slot $s_j^*$ which maximizes her utility (greedy bidding choice), that is,*

$$s_j^* = \operatorname{argmax}\{\theta_s(v_j - p_{-j}(s)) : s \geq s_j\}.$$

- *chooses her next bid as $b' = \min\{v_j, p_{-j}(s_{j-1}^*) - \varepsilon_{price}\}$.*

Convergence issues for CB are much more serious than for BB: in general, the CB strategy does not have a fixed point!

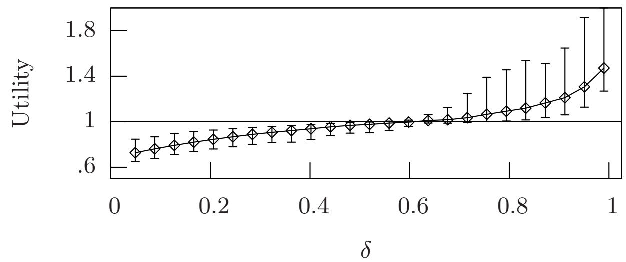


**Figure 3: RBB vs. BB Player Utility (BB=1)**

Indeed, a player $p$ in slot $i$ will only have non-negligible utility if the player in slot $i+1$ cannot raise her bid any further, because it is equal to her value. In this case, unless the situation is degenerate, $p$ will prefer to move to a different slot, which will in turn lead other players to want different slots.

Thus, the only fixed point for CB is when all players are bidding their values and those bids happen to be a Nash equilibrium for the GSP strategy. Bids equal to the values form a Nash equilibrium if and only if $\theta_i(v_i - v_{i+1}) \geq \theta_j(v_i - v_{j+1})$ for each $i$ and $j > i$; for a random instance, these constraints are satisfied with significant probability when $\delta$ is close to 0.

Indeed, Figure 4 confirms this finding. Like in Figure 2, we plot two curves, which are lower and upper bounds on the probability of convergence. (The space between the two curves corresponds to instances that have neither converged nor begun to cycle). We see that CB frequently converges when $\delta$ is small, and rarely converges when $\delta$ is large.

Given that CB is a well-known strategy, the auctioneer might wonder how much revenue he obtains if every player followed the CB strategy. Figure 5 compares the revenue obtained from running GSP with the players following CB, with the revenue which would have been obtained from running VCG (with VCG revenue normalized to 1). This is good news for the auctioneer: *If the players follow the* CB *strategy, then the auctioneer's revenue is higher than the VCG revenue, by up to* 20%. Accordingly, player utility goes down.

### 5.4.2 Altruistic Bidding

Since competitor busting primarily benefits the auctioneer, the players might consider trying a completely different approach: altruism. A natural complement to the competitor busting strategy, where players try to hurt other bidders as much as possible, the altruistic strategy has players bidding as low as possible to win their desired slot.

**Definition** 15. *The* Altruistic *greedy strategy (*AB*), is the strategy for a player $j$ that, given $b_{-j}$*

- *next targets the slot $s_j^*$ which maximizes her utility (greedy bidding choice), that is,*

$$s_j^* = \text{argmax}\{\theta_s(v_j - p_{-j}(s)) : s \geq s_j\}.$$

- *chooses her next bid as $b' = \min\{v_j, p_{-j}(s_j^*) + \varepsilon_{price}\}$. If there is no slot giving positive utility, bid $v_j$.*

Convergence issues for AB are even more serious than for CB: the AB strategy has *no* fixed points when the $\theta_i$'s are separated.

The auctioneer might worry about how much revenue he would obtain if every player followed the AB strategy. As can be seen in Figure 6, there is indeed cause for worry: AB *produces very low revenue, much less than the VCG revenue,* where again VCG revenue is normalized to 1. It would be interesting to see if there are any auctions in practice where players were bidding in this way.

## 5.5 Theoretical results

The following Theorem compares the auctioneer revenue obtainable by the GSP mechanism to the revenue which could have been obtained by using the VCG mechanism instead.

**Theorem** 16. *1. For every $K > 0$, there exists a keyword auction and a Nash equilibrium of the GSP mechanism whose revenue is at most $1/K$ times the revenue of the VCG mechanism, and moreover, every bidder $i$ bids $b_i \leq v_i$.*

*2. For every $K > 0$, there exists a keyword auction and a Nash equilibrium of the GSP mechanism whose revenue is at least $K$ times the revenue of the VCG mechanism.*

*3. If every bidder $i$ bids $b_i \leq v_i$ (ie. is debt-averse), then for every keyword auction and Nash equilibrium of the GSP mechanism, the revenue is at most $\alpha^*$ times the VCG revenue, were $\alpha^* = \max_i \theta_i/(\theta_i - \theta_{i+1})$.*

PROOF. *Part 1:* Here is an example showing that the revenue to the auctioneer from a GSP equilibrium may be arbitrarily smaller than that the revenue from the VCG equilibrium. The example is for a two-slot auction, where $\theta_1 = 1$ and $\theta_2 = \frac{1}{2}$. The GSP revenue is $2\theta_1 + (1/2)\theta_2 = 2.25$ whereas the VCG revenue is $(x+1/2)\theta_1 + (1/2)\theta_2 = x + (3/4)$.

A GSP equilibrium much smaller than VCG

| player | value | bid | slot won | GSP pr. | VCG pr. |
|---|---|---|---|---|---|
| 1 | $x+1$ | $x+1$ | 1 | 2 | $x+\frac{1}{2}$ |
| 2 | $x$ | 2 | 2 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3 | 1 | 1 | – | 0 | 0 |

*Part 2:* Here is an example showing that the revenue to the auctioneer from a GSP equilibrium may be arbitrarily larger than that the revenue from the VCG equilibrium. The example is for a two-slot auction, where $\theta_1 = 1$ and $\theta_2 = \frac{1}{2}$.

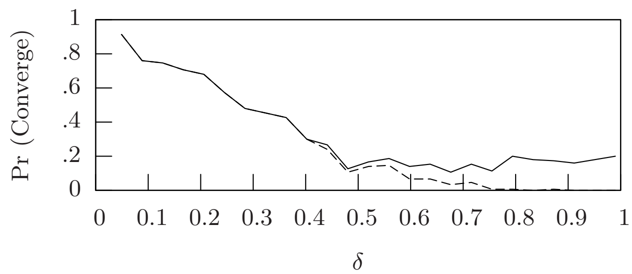


**Figure 4: Convergence of CB**

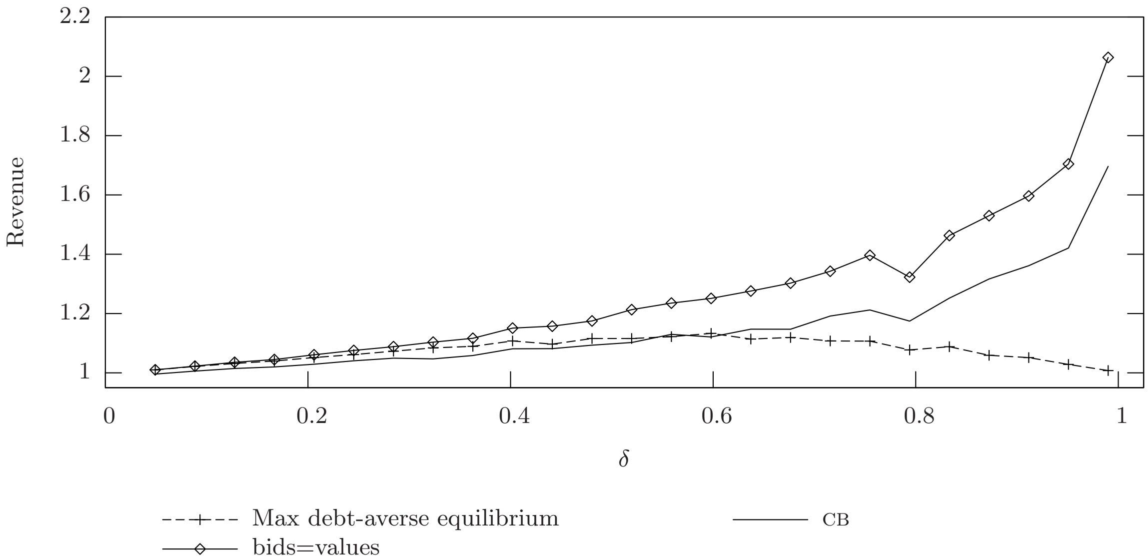


**Figure 5: Comparing revenue from CB with the VCG revenue (VCG=1)**

A GSP equilibrium much larger than VCG

| player | value | bid | slot won | GSP pr. | VCG pr. |
|---|---|---|---|---|---|
| 1 | $x$ | $x$ | 1 | $(x+2)/2$ | $\frac{3}{2}$ |
| 2 | 2 | $\frac{(x+2)}{2}$ | 2 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3 | 1 | 1 | – | 0 | 0 |

These examples rely on player values that are arbitrarily separated. The second example was constructed using the following expression due to Varian [11], which gives the payments that achieve the maximum Nash revenue.

$$\theta_i p_i = \sum_{j>i} v_{j-1}(\theta_j - \theta_{j+1}) \quad (3)$$

Compare this with the VCG payments:

$$\theta_i p_i = \sum_{j>i} v_j(\theta_j - \theta_{j+1}) \quad (4)$$

Observe that if the values are close to each other, the maximum revenue from a Nash equilibrium is close to the VCG revenue. For example, if $v_{i+1} \geq \alpha v_i$, then the payments from (3) are at most a factor of $1/\alpha$ from the payments of (4).

Note that the second example showing a Nash revenue larger the VCG revenue has the second player bidding much more than his value and so is not debt-averse as we defined in Section 5.2. Under the more realistic debt-averse assumption we have the third part of this theorem.

*Part 3:* Let $R^M$ be the maximum debt-averse Nash revenue. Note that the payment of $i$, $p_i^M$, is at most $v_{i+1}$. If $p_i^M > v_{i+1}$, as all players are bidding at most their values, there would not be enough winners to fill the top $i$ slots. Thus the maximum risk-free Nash equilibrium revenue is $R^M \geq \sum_{1\leq i\leq k} \theta_i v_{i+1}$. On the other hand, the VCG payment of $i$ is $\theta_i p_i^{\mathrm{VCG}} \geq (\theta_i - \theta_{i+1})v_{i+1}$. Thus the total VCG revenue is at most $\sum \theta_i p_i^{\mathrm{VCG}} \geq \sum(\theta_i - \theta_{i+1})v_{i+1} = \sum(\theta_i - \theta_{i+1})\theta_i v_{i+1}/\theta_i \geq R^M/\alpha^*$ as required. □

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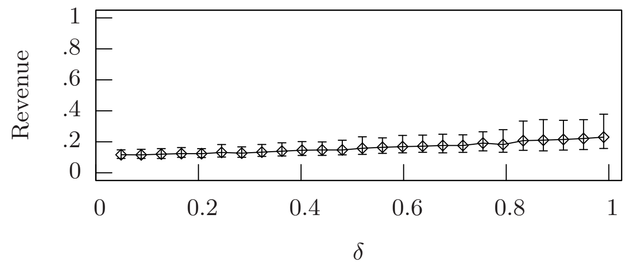


**Figure 6: AB total revenue (VCG=1)**


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# APPENDIX

## Proof Sketch of Theorem 6, Part 3.

At time $t$, we say that a player $p$ is *activated* if $p$ is the player who updates his bid while the other bidders repeat their previous bids.

**Lemma** 17. *Let $T$ be a certain function of $n, k, (\theta_i), (v_j)$. For every starting configuration, there exists a sequence of player activations, of length at most $T$, such that the resulting configuration is a fixed point.*

Proof. Consider an arbitrary starting configuration. To construct the sequence, the idea is to emulate the proof of Theorem 10.

First, we emulate the proof of Lemma 12 as follows. Repeatedly activate all the players $1, \ldots, k+1$ until each of them bids at least $v_{k+1}$. In other words, if one of the players in $[1, k]$ have a current bid $< v_{k+1}$, we activate all the players $1, \ldots, k+1$ one at a time. By Lemma 12 after a total of at most $k+1)t_1$ activations all players $[1, k+1]$ bid at least $v_{k+1}$, where $t_1$ as defined in Lemma 12. Now if there are some players in $[k+2, n]$ who are not bidding their value, activate one of these players so that will now bid their value. The conclusion of Lemma 12,

$$\begin{cases} b_i & > & v_{k+1} & \forall i \leq k \\ b_i & = & v_i & \forall i \geq k+1 \end{cases}$$

will hold after at most $(k+1)t_1 + n - k + 1$ activations. From that point on until the end of the sequence, players $[k+1, n]$ will not be activated again. The rest of the sequence is partitioned into phases, corresponding to stable sets, defined as in the proof of Theorem 10. A stable set stays the same throughout a phase. To define the sequence during a phase, let $A$ be the current stable set, $[i+1, k]$ be the slots occupied by the player of $A$, and $B$ be the set of players occupying slots $[1, i]$. Let $p_{\min}$ be the player in $B$ whose value is minimum, $v_{\min}$ be its value, and $p_{i-1}$ be the price of slot $i-1$.

Consider the three cases enumerated in the proof of Theorem 10. We repeatedly activate the player currently in slot $i$ until either Case 1 or Case 2 occurs, or $p_{i-1} > v_{\min} - \varepsilon$, where $\varepsilon$ is defined as in Lemma 13. We then activate player $p_{\min}$. At this point, Case 1 or Case 2 must occur, a new stable set is defined, and the phase ends. This completes the definition of the sequence.

We now need to bound the length of the constructed sequence of player activations, independently of the starting configuration and bids.

In the initial part players $[1, k+1]$ are activated at most $t_1$ times and players $[k+2, n]$ are activated at most once. Thus this part has length at most $(k+1)t_1 + n - k + 1$.

It is also easy to see, as done in the proof of Theorem 10, that the stable set in the next phase will be larger in the $\succ$ order, hence there will be at most $2^k$ phases, ending with a fixed point.

To bound the length of a phase, it is again easy to see that during a phase, $b_{\min}$ can only increase; moreover, when a player $p$ is activated for several times, the new value of $b'_{\min}$ must be larger than the value of $b_{\min}$ during the time of his previous activation by at least $b'_{\min} \geq b_{\min} + (1 - \gamma^*)(v_p - b_{\min})$, precisely the same inequality as in the analysis of Case 3 in the proof of Theorem 10. Since we must have repetitions of choices of player at least once every $k$ activations, if we define $x$ as in proof of Theorem 10, it follows that after at most $xk$ activations we are ready to activate $p_{\min}$ and end the phase.

Hence the length $T$ of the sequence is at most $n + 3k$ plus $k$ times the bound in Theorem 10, Part 2, which is independent of the starting configuration.

□

With this Lemma, it is easy to complete the proof of Theorem 6: In a random sequence, at every step we have probability at least $1/n$ to choose the next activation as in Lemma 17, hence the sequence will occur after about $n^T$ steps on average, and in any case, it will occur with probability 1 after a finite time. This proves convergence.