The VLSI Model III

Planar Separator Theorem

**Theorem** Let $G = (V,E)$ be an $N$-vertex planar graph having non-negative vertex costs summing to $c(V)$. Then, $V$ can be partitioned into three sets, $A$, $B$, and $C$, such that no edge joins vertices in $A$ with those in $B$, neither $A$ nor $B$ has cost exceeding $2c(V)/3$, and $C$ contains no more than $4\sqrt{N}$ vertices.

**Proof** We assume $G$ connected. If not, embed it in the plane and add edges as appropriate to make it connected. Assume that it has been triangulated, that is, every face except for the outermost is a triangle.

Pick any vertex (call it the root) and perform a breadth-first traversal of $G$. This traversal defines a **BFS spanning tree** $T$ of $G$.

Vertices in $G$ are partitioned into the following five sets: a) $H = \bigcup_{d < l} R_d$ (high vertices close to the root), b) $R_l$ (vertices at level $l$), c) $M = \bigcup_{l < d < h} R_d$ (middle vertices), d) $R_h$, e) $L = \bigcup_{h < d} R_d$ (low vertices).

Since $L$ and $H$ are subsets of the sets of vertices with levels less than and more than $m$, $c(L), c(H) \leq c(V)/2$. Also, by construction, $r_l, r_h \leq \sqrt{N}$.

If $R_l = R_h = R_m$ (which implies that $M$ is empty), let $A = L$, $B = H$, and $C = R_l = R_h$. Then, $C$ is a separator of size at most $\sqrt{N}$ and the theorem holds. If $l \neq h$, then $h - l - 1 \geq 0$. Since each of the $h - l - 1$ levels between $r_l$ and $r_h$ has at least $\sqrt{N} + 1$ vertices, it follows that $h - l - 1 \leq \sqrt{N} - 1$ because these levels have $\leq N - 1$ vertices.
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Consider the subgraph of $G$ consisting of the vertices in $M$ and the edges between them. Add a new vertex $v_0$ to replace the vertices in $H \cup R_I$ and add an edge from $v_0$ to each of the vertices at level $I + 1$. This operation retains planarity and the resulting graph remains triangulated because adjacent vertices on $R_{I+1}$ have an edge between them. Also, it defines a spanning tree $T^*$ consisting of $v_0$, the new edges, and the projection of the original spanning tree to the vertices in $M$. $T^*$ has radius at most $\sqrt{N}$.

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Apply Lemma of last lecture to $T^*$ while giving $v_0$ zero cost. The lemma identifies three sets of vertices, $A_0$, $B_0$, and $C_0$, from which we delete $v_0$. Since $c(M) \leq c(V)$, it follows that there are no edges between vertices in $A_0$ and $B_0$, $c(A_0) = c(B_0) \leq 2c(V)/3$, and $|C_0| \leq 2 \sqrt{N}$. Let $C = C_0 \cup R_I \cup R_{I'}$. Thus, $|C| \leq 4 \sqrt{N}$.

Each of the four sets $A_0$, $B_0$, $L$, and $H$ has cost at most $2c(V)/3$. If any one of them has cost more than $c(V)/3$, let it be $A$; let $B$ be the union of remaining sets. If none of them has cost more than $c(V)/3$ vertices, order the sets by size and let $A$ be the union of the fewest of these sets whose cost is at least $c(V)/3$ vertices. This procedure ensures that $A$ has cost between $c(V)/3$ and $2c(V)/3$ which implies that $B$ satisfies the same condition as $A$ and theorem is established. QED

Planar Separator Theorem

Lemma Let $G = (V,E)$ be an $N$-vertex planar graph having non-negative vertex costs summing to $c(V)$. Then $V$ can be partitioned into three sets, $A$, $B$, and $C$, such that no edge joins vertices in $A$ with those in $B$, neither $A$ nor $B$ has cost exceeding $7c(V)/9$, $|A|$, $|B| \leq 5N/6$, and $C$ contains no more than $K_I \sqrt{N}$ vertices, where $K_I = 4(\sqrt{2/3} + 1)$.

Lemma Let $G = (V,E)$ be an $N$-vertex planar graph and let $c$ be a non-negative cost function on $V$ with total cost of $c(V)$. Let $P \geq 2$. There are constants $2P/3 \leq q \leq 3P$ and $K_2 = 4(\sqrt{2/3} + 1)/(1 - \sqrt{5/6})$ such that $V$ can be partitioned into $q$ sets, $A_1$, $A_2$, $\ldots$, $A_q$ such that for $1 \leq i \leq q$

\[ c(V)/(3P) \leq c(A_i) \leq 3c(V)/(2P) \]

and there are sets $C_i$, $|C_i| \leq K_2 \sqrt{N}$, and $B_i = V - A_i - C_i$ such that no edges join vertices in $A_i$ with vertices in $B_i$. QED