Overview

- Completion of Planar Separator Theorem
Area-Time Computational Inequalities

\[ C_p(f) = O(min(AT^2, A^2T)) \]

- To derive lower bounds on \( C_p(f) \) we introduce the \textit{planar separator theorem}. In its simplest form, it states that the vertices in every planar \( n \)-vertex graph can be divided into two sets with no edges between them by the removal of \( O(\sqrt{n}) \) vertices such that each set has between \( n/3 \) and \( 2n/3 \) vertices.

- We use it to show that some functions have a quadratic planar circuit size in their number of inputs. The technique used is to show that a lot of information must pass from inputs to outputs.
Planar Separator Theorem

- Let $G = (V,E)$ and let $c : V \rightarrow R$ assign non-negative costs to vertices. The cost of a subset $S$ of $V$ is the sum of the cost of the elements of $S$.

**Lemma** If $G$ has a rooted spanning tree of radius $r$, $V$ can be partitioned into disjoint sets $A$, $B$, $C$ such that $c(A), c(B) \leq 2c(V)/3$, no edge joins vertices in $A$ and $B$, and $C$ contains at most $2r+1$ vertices.
Planar Separator Theorem

**Theorem I** Let $G = (V,E)$ be an $N$-vertex planar graph having non-negative vertex costs summing to $c(V)$. Then, $V$ can be partitioned into three sets, $A$, $B$, and $C$, such that no edge joins vertices in $A$ with those in $B$, neither $A$ nor $B$ has cost exceeding $2c(V)/3$, and $C$ contains no more than $4\sqrt{N}$ vertices.
Planar Separator Theorem

Proof We assume \( G \) is connected. If not, embed it in the plane and add edges as appropriate to make it connected. Assume that it has been triangulated. Pick any vertex (call it the root) and perform a breadth-first traversal of \( G \). This traversal defines a BFS spanning tree \( T \) of \( G \).
Planar Separator Theorem

A vertex $v$ has level $d$ in this tree if the length of the path from the root to $v$ has $d$ edges. There are no vertices at level $q$ where $q$ is the level one larger than that of all vertices ($q = 3$ in the example). Let $R_d$ be the vertices at level $d$ and let $r_d = |R_d|$. 
Planar Separator Theorem

In Problem 12.9 it is stated that there is some level $m$ such that the cost of vertices at levels below and above $m$ each is at most $c(V)/2$. Let $l$ and $h$, $l \leq m \leq h$, be levels closest to $m$ that contain at most $\sqrt{N}$ vertices. That is, $r_l, r_h \leq \sqrt{N}$. There are such levels because level 0 contains a single vertex and there are none at level $q$. 
Planar Separator Theorem

Vertices in G are partitioned into five sets: a) $H = \bigcup_{d < l} R_d$ (high vertices close to the root), b) $R_l$ (vertices at level $l$), c) $M = \bigcup_{l < d < h} R_d$ (middle vertices), d) $R_h$, e) $L = \bigcup_{h < d} R_d$ (low vertices).

Because $L$ and $H$ are subsets of the vertices with levels less than and more than $m$, $c(L)$, $c(H) \leq c(V)/2$. By construction, $r_l, r_h \leq \sqrt{N}$. 
Planar Separator Theorem

If $R_l = R_h = R_m$ (which implies that $M$ is empty), let $A = L$, $B = H$, and $C = R_l = R_h$. Then, $C$ is a separator of size at most $\sqrt{N}$ and the theorem holds. If $l \neq h$, then $h - l - 1 \geq 0$.

Since each of the $h - l - 1$ levels between $r_l$ and $r_h$ has at least $\sqrt{N} + 1$ vertices, it follows that $h - l - 1 \leq \sqrt{N} - 1$ because these levels have at most $N - 1$ vertices.
Planar Separator Theorem

Consider the subgraph of $G$ consisting of the vertices in $M$ and the edges between them. Add a new vertex $v_0$ to replace the vertices in $H \cup R_i$ and add an edge from $v_0$ to each of the vertices at level $l+1$. This operation retains planarity and the resulting graph remains triangulated because adjacent vertices on $R_{i+1}$ have an edge between them. Also, it defines a spanning tree $T^*$ consisting of $v_0$, the new edges, and the projection of the original spanning tree to the vertices in $M$. $T^*$ has radius at most $\sqrt{N}$. 
Planar Separator Theorem

Apply Lemma of last lecture to $T^*$ while giving $v_0$ zero cost. The lemma identifies three sets of vertices, $A_0$, $B_0$ and $C_0$, from which we delete $v_0$. Since $c(M) \leq c(V)$, it follows that there are no edges between vertices in $A_0$ and $B_0$, $c(A_0), c(B_0) \leq 2c(V)/3$, and $|C_0| \leq 2 \sqrt{N}$. Let $C = C_0 \cup R_l \cup R_h$. Thus, $|C| \leq 4 \sqrt{N}$. 
Planar Separator Theorem

Each of the four sets \(A_0, B_0, L,\) and \(H\) has cost at most \(2c(V)/3\). If any one of them has cost more than \(c(V)/3\), let it be \(A\); let \(B\) be the union of remaining sets. It follows that \(c(V)/3 \leq c(A),\) \(c(B) \leq 2c(V)/3.\)

If none of them has cost more than \(c(V)/3\), order the sets by size and let \(A\) be the union of the fewest of these sets whose cost is \(\geq c(V)/3\) vertices. This procedure ensures that \(c(V)/3 \leq c(A) \leq 2c(V)/3\) which implies that \(B\) satisfies the same condition and theorem is established. QED
Planar Separator Theorem

**Theorem II** Let $G = (V,E)$ be an $N$-vertex planar graph with non-negative vertex costs summing to $c(V)$. Then $V$ can be partitioned into three sets, $A$, $B$, and $C$, such that no edge joins vertices in $A$ with those in $B$, neither $A$ nor $B$ has cost exceeding $7c(V)/9$, $|A|, |B| \leq 5N/6$, and $C$ contains no more than $K_1 \sqrt{N}$ vertices, where $K_1 = 4(\sqrt{(2/3)} + 1)$. 
Planar Separator Theorem

Theorem III Let $G = (V,E)$ be an $N$-vertex planar graph and let $c$ be a non-negative cost function on $V$ with total cost of $c(V)$. Let $P \geq 2$. There are constants $2P/3 \leq q \leq 3P$ and $K_2 = 4(\sqrt{(2/3)} + 1)/(1 - \sqrt{(5/6)})$ such that $V$ can be partitioned into $q$ sets, $A_1, A_2, ..., A_q$ such that for $1 \leq i \leq q$

$$c(V)/(3P) \leq c(A_i) \leq 3c(V)/(2P)$$

and sets $C_i$, $|C_i| \leq K_2 \sqrt{N}$, and $B_i = V - A_i - C_i$ such that no edges join vertices in $A_i$ with vertices in $B_i$. 