Hong-Kung Lower Bound

Definition S-span of DAG G, \( \rho(S,G) \), is the max number of vertices of G that can be pebbled with S red pebbles in red pebble game maximized over all initial placements of S red pebbles.

Theorem For every pebbling \( \mathcal{P} \) of \( G = (V,E) \) in the red-blue pebble game with S red pebbles, the I/O time used, \( T_2^{(2)}(S,G,\mathcal{P}) \) satisfies
\[
\left[ T_2^{(2)}(S,G,\mathcal{P})/S \right] \rho(2S,G) \geq |V| - |In(G)|
\]

The Fast Fourier Transform

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Lemma The S-span of the FFT graph \( F^{(d)} \) on \( n = 2^d \) inputs satisfies \( \rho(S,F^{(d)}) \leq 2S \log S \) when \( S \leq n \).

Proof Let \( \{p_1, ..., p_S\} \) be pebbles used in game. We use \( \text{num}(p_i) (= 0 \text{ initially}) \) to over bound the S-span. \( F^{(d)} \) contains many 2-input FFT (butterfly) graphs. If \( v_j \) about to be pebbled, \( p_1 \) and \( p_2 \) must be on \( u_1 \) and \( u_2 \). Upper bound on S-span assumes \( v_2 \) also pebbled. That is, we advance pebbles to \( v_1 \) and \( v_2 \) (even though this violates the rules). If \( \text{num}(p_j) = \text{num}(p_2) \), increase each by 1. If not, increase smaller by 1. Because 2 pebble placements each time \( \text{num}(p_i) \) increases, \( \rho(S,F^{(d)}) \leq 2(\text{num}(p_1) + ... + \text{num}(p_S)) \). We show that \( \text{num}(p_i) \leq \log_2 S \).
The Fast Fourier Transform

Proof (cont.) Let \( p_i \) reside on \( v_i \). Let \( N(i) \) be the number of initially pebbled predecessors of \( v_i \). By induction we show that \( N(i) \geq 2^a \), \( a = \text{num}(p_i) \).

Base case: \( \text{num}(p_i) = 1 \) corresponds to first move of \( p_i \) to a vertex with 2 initially pebbled predecessors.

Inductive hypothesis: \( N(i) \geq 2^a \) for \( a = \text{num}(p_i) \leq e-1 \).

Consider first point in time at which \( \text{num}(p_i) = e \). Let \( p_i \) move to \( v_i \). At previous step \( p_i \) and second pebble \( p_j \) reside on predecessors \( u_1 \) and \( u_2 \) of \( v_i \). Either a) \( \text{num}(p_i) = \text{num}(p_j) = e-1 \) or b) \( \text{num}(p_i) < \text{num}(p_j) \).

In a) since \( u_1 \) and \( u_2 \) are roots of disjoint trees, the number of predecessors of \( u_1 \) and \( u_2 \) carrying pebbles initially is at least \( 2(2^{e-1}) = 2^e \). In b), \( u_1 \) has at least \( 2^{e-1} \) predecessors with pebbles and \( u_2 \) has at least \( 2^e \).

Since at most \( S \) pebbles are on the graph initially, \( N(i) \leq S \), \( \text{num}(p_i) \leq \log_2 S \) & \( \rho(S,F^{(d)}) \leq 2S \log_2 S \). QED

The Fast Fourier Transform

Theorem

When \( F^{(d)} \) pebbled in the red-blue pebble game with \( S \geq 3 \), the following hold simultaneously:

\[
T_1^{(2)}(S,F^{(d)},P) = \Theta(n \log n) \\
T_2^{(2)}(S,F^{(d)},P) = \Theta(n \log n/ \log S)
\]

Proof

First lower bound is obvious. 2nd follows from S-span bound. Let \( \beta = 2^e \). For upper bounds, consider decomposition into copies of \( F^{(e)} \) on \( \beta \) inputs. Each vertex in \( F^{(e)} \) pebbled once with \( \beta+1 \) red pebbles.

Blues are only used on inputs and outputs to \( F^{(e)} \).