Overview

• Application of the Hong-Kung Bound to the Fast Fourier Transform
Fast Fourier Transform

**Definition** The S-span of DAG G, \( \rho(S, G) \), is the maximum number of vertices of G that can be pebbled with S red pebbles in red pebble game maximized over all initial placements of S red pebbles. (Initialization rule is disallowed.)

**Theorem** For every pebbling P of G = (V,E) in the red-blue pebble game with S red pebbles, the I/O time used, \( T_2^{(2)}(S, G, P) \) satisfies

\[
\left\lfloor \frac{T_2^{(2)}(S, G, P)}{S} \right\rfloor \rho(2S, G) \geq |V| - |\text{In}(G)|
\]
Fast Fourier Transform (FFT)

- We derive matching upper and lower bounds on I/O complexity and number of computation steps to pebble the FFT graph $F^{(d)}$ on $n = 2^d$ inputs with $S$ red pebbles.
Fast Fourier Transform

**Lemma** The S-span of the FFT graph $F^{(d)}$ on $n = 2^d$ inputs satisfies $\rho(S, F^{(d)}) \leq 2S \log_2 S$ when $S \leq n$.

**Proof** Let \{p_1, ..., p_S\} be pebbles used in the game. We use $\text{num}(p_i)$ (= 0 initially) to overbound the S-span. $F^{(d)}$ contains many 2-input FFT (butterfly) graphs.
Fast Fourier Transform

If \( v_1 \) is about to be pebbled, \( p_1 \) and \( p_2 \) must be on \( u_1 \) and \( u_2 \). Upper bound on \( S \)-span assumes \( v_2 \) also pebbled. That is, we advance pebbles to \( v_1 \) and \( v_2 \) (even though this violates the rules). If \( \text{num}(p_1) = \text{num}(p_2) \), increase each by 1. If not, increase smaller by 1. Because there are two pebble placements each time \( \text{num}(p_i) \) increases, \( r(S,F^{(d)}) \leq 2(\text{num}(p_1) + \ldots + \text{num}(p_S)) \). We show that \( \text{num}(p_i) \leq \log_2 S \).
Fast Fourier Transform

Proof (cont.) Let \( p_i \) reside on \( v_i \). Let \( N(i) \) be the number of initially pebbled predecessors of \( v_i \). By induction we show that \( N(i) \geq 2^a \), \( a = \text{num}(p_i) \).

Base case: \( \text{num}(p_i) = 1 \) corresponds to first move of \( p_i \) to a vertex with two initially pebbled predecessors.

Inductive hypothesis: \( N(i) \geq 2^a \) for \( a = \text{num}(p_i) \leq e-1 \). Consider first point in time at which \( \text{num}(p_i) = e \). Let \( p_i \) move to \( v_i \). At previous step \( p_i \) and second pebble \( p_j \) reside on predecessors \( u_1 \) and \( u_2 \) of \( v_i \). Either a) \( \text{num}(p_i) = \text{num}(p_j) = e-1 \) or b) \( \text{num}(p_i) < \text{num}(p_j) \). In a) since \( u_1 \) and \( u_2 \) are roots of disjoint trees, the no. of predecessors of \( u_1 \) and \( u_2 \) carrying pebbles initially is at least \( 2(2^{e-1}) = 2^e \). In b), \( u_1 \) has at least \( 2^{e-1} \) predecessors with pebbles and \( u_2 \) has at least \( 2^e \). Since at most \( S \) pebbles are on the graph initially, \( N(i) \leq S \), \( \text{num}(p_i) \leq \log_2 S \) & \( \rho(S,F(d)) \leq 2S\log_2 S \) QED
Fast Fourier Transform

Figure 11.9 Decomposition of the FFT graph $F^{(d)}$ into $\beta = 2^e$ bottom FFT graphs $F^{(d-e)}$ and $\tau = 2^{d-e}$ top $F^{(e)}$. Edges between bottom and top sub-FFT graphs identify common vertices between the two.
Fast Fourier Transform

**Theorem** When $F^{(d)}$ pebbled in the red-blue pebble game with $S \geq 3$, the following hold simultaneously:

$$T_1^{(2)}(S, F^{(d)}, \mathcal{P}) = \Theta(n \log n)$$
$$T_2^{(2)}(S, F^{(d)}, \mathcal{P}) = \Theta(n \log n / \log S)$$

**Proof** First lower bound is obvious. 2nd follows from $S$-span bound. Let $\beta = 2^e$. For upper bounds, consider decomposition into copies of $F^{(e)}$ on $\beta$ inputs. Each vertex in $F^{(e)}$ is pebbled once with $\beta + 1$ red pebbles. Blues are only used on inputs and outputs to $F^{(e)}$. 
Fast Fourier Transform

Let $e = \lceil \log_2 (S-1) \rceil$ or $S \geq \beta + 1$ Then, we pebble all vertices in the top FFT graphs $F^{(e)}$ with blue pebbles only on inputs and outputs. We decompose each of the bottom FFT $F^{(d-e)}$ graphs into graphs $F^{(e)}$ at the top with $F^{(d-e)}$ graphs on the bottom and pebble the graphs $F^{(e)}$ in the same way. Thus, we pebble with blue pebbles the vertices at levels in $F^{(d)}$ separated by $\log_2 (S-1)$ levels. Since there are $(\log_2 n + 1)$ levels and $n$ vertices per level, this gives the upper bound on $T_2^{(2)}(S,F^{(d)},P)$. The upper bound on $T_1^{(2)}(S,F^{(d)},P)$ follows because each vertex is pebbled once with a red pebble. QED