Overview

- Slice Functions
- Pseudo-negation
- Monotone circuit size of slice functions
- Half-Clique Central Slice – an NP-complete problem
Slice Functions

- Some monotone functions have exponential circuit size over the monotone basis, doubtful that same methods of analysis can be extended to derive such bounds over the standard basis.
- We show that the monotone circuit size of monotone slice functions can provide a strong lower bound on the circuit size of such functions over the standard basis. Also, there are $\text{NP}$-complete languages whose characteristic functions are slice functions. Thus, if such functions can be shown to have super-polynomial monotone circuit size, $\text{P} \neq \text{NP}$.
Slice Functions

- Let $|x|$ be no. 1’s in $x$. A **slice function** $s(x)$ has value 0 for $|x| < k$ & value 1 for $|x| > k$ for some $k$.
- For $f : B^n \rightarrow B$, $f^{[k]}$ is the **$k$ th slice function** if it is 0 for $|x| < k$, 1 for $|x| > k$, and equal to $f$ otherwise.
- Note that slice functions are monotone!
Slice Functions

Lemma For $f : B^n \rightarrow B$ over the standard basis

$$C(f) = C(f[0], f[1], ..., f[n]) + O(n)$$

Proof Given $f[0], f[1], ..., f[n]$ we construct a circuit for $f$ from the multiplexer function and a circuit to count the number of 1s in the input. Supply output of the counting circuit to input of the multiplexer along with the outputs of the circuits for $f[0], f[1], ..., f[n]$. 
Slice Functions

- The following result says that for every Boolean function there is some slice of it that has a circuit size close to the circuit size of the entire function when its circuit size is large.

**Theorem** For $f : B^n \rightarrow B$ over the standard basis there exists $k$ such that

$$\frac{C(f)}{n} - O(1) \leq C(f^k) \leq C(f) + O(n)$$
Slice Functions

**Theorem** For $f : B^n \rightarrow B$ over the standard basis there exists $k$ such that

$$\frac{C(f)}{n} - O(1) \leq C(f^{[k]}) \leq C(f) + O(n)$$

**Proof** Because $C(f^{[0]}, f^{[1]}, \ldots, f^{[n]}) \leq \Sigma C(f^{[i]})$, there must be $i = k$ for which $C(f^{[i]})$ is largest. The lower bound follows from this. The upper bound follows from $f^{[k]}(x) = (\tau^{(n)}_k(x) \land f(x)) \lor \tau^{(n)}_{k+1}(x)$ where $\tau^{(n)}_k$ is the threshold function with threshold $k$. 
Pseudo-Negation

**Definition** A *pseudo-negation* for a variable $x_i$ in a function $f$ is a function $h$ such that replacing each instance of $x_i$ by $h$ does not change the value of $f$.

The *punctured threshold function* $\tau^{(n)}_{k,i}$ is $\tau^{(n-1)}_k$ applied to all variables except $x_i$.

Since $\tau^{(n)}_{k,i}$ can be realized by a binary sorter, $C_{\text{mon}}(\tau^{(n)}_{k,i})$ is $O(n \log n)$. Thus, all $n$ punctured threshold functions on $n$ variables can be realized by $O(n^2 \log n)$ gates over the monotone basis. This bound can be improved to $O(n \log^2 n)$. 
Monotone Circuit Size of Slice Functions

The following theorem shows that if the monotone circuit size of a slice function is large, its standard circuit size is also large.

**Theorem** Let $f : B^n \to B$ be a slice function and $C_{\text{mon}}(f)$ and $C(f)$ be its circuit size over the monotone and standard bases. Then,

$$C(f) \leq C_{\text{mon}}(f) \leq 2C(f) + O(n \log^2 n)$$
Monotone Circuit Size of Slice Functions

Proof Convert an optimal circuit for $f$ to dual-rail logic. This at most doubles the number of gates and has as inputs $x_i$ and $x_{\bar{i}}$.

Let $k =$ threshold of $f$. $f = 0 \ (1)$ if $|x| < k \ (> k)$.

$$\tau^{(n)}_{k,i} = 0 \ (1) \text{ when } |x| < k \ (> k).$$

When $|x| = k$, $\tau^{(n)}_{k,i} = 0 \ (1)$ if $x_i = 1 \ (0)$. 
Monotone Circuit Size of Slice Functions

Replace $\bar{x}_i$ by $\tau^{(n)}_{k,i}$. When $|x| < k$, $f = 0$ whether $x_i = 0$ or 1. Replacing $\bar{x}_i$ by $\tau^{(n)}_{k,i} = 0$ doesn’t change $f$.

When $|x| > k$, $f = 1$ whether $x_i = 0$ or 1. Replacing $\bar{x}_i$ by $\tau^{(n)}_{k,i} = 1$ doesn’t change $f$.

Finally, when $|x| = k$, $\tau^{(n)}_{k,i}$ behaves like $\bar{x}_i$. 
Central Slice

Definition The **central slice** of \( f : \mathbb{B}^n \rightarrow \mathbb{B} \), is \( f[k] \) for \( k = \lceil n/2 \rceil \).

Definition The **central clique function** \( f_{\text{clique},n/2}^{(n)} \) on \( n(n-1)/2 \) inputs (denoting presence or absence of edges in a graph on \( n \) vertices) has value 1 if the graph has a clique on \( \lceil n/2 \rceil \) vertices and 0 otherwise.

Let \( e(k) = k(k-1)/2 \) be number of edges in a \( k \)-clique. The central slice of \( f_{\text{clique},n/2}^{(n)} \) is called the **half clique central slice function** \( f_{\text{clique slice}}^{(n)} \). It has value 1 if the graph denoted by inputs has a clique on \( \lceil n/2 \rceil \) inputs or it has more than \( e(n/2) \) edges.
Central Slice

HALF CLIQUE CENTRAL SLICE

*Input*: Undirected graph $G = (V, E)$, $|V|$ even

*Answer*: “Yes” if $G$ has a clique on $|V|/2$ vertices or it has at least $e(|V|/2)$ edges.
Half Clique Central Slice

- The following theorem demonstrates that the half clique central slice function has a monotone circuit size at least as large as every slice function of the central clique function. Combined with earlier results, this demonstrates that the central slice of the half-clique function on \( n/2 \) vertices has polynomial-size circuits if and only if the half-clique function does.

**Theorem** HALF CLIQUE CENTRAL SLICE is \( \text{NP} \)-complete. Furthermore, for all \( 1 < k < n \),

\[
C_{\text{mon}}((f_{\text{clique},n/2}^{(n)})[k]) \leq C_{\text{mon}}(f_{\text{clique slice}}^{(n)})
\]
Half Clique Central Slice

**Proof** Reduce from instance $G = (V,E)$ with $n$ vertices, $n$ even, of HALF-CLIQUE to instance $G' = (V',E')$ of HALF CLIQUE CENTRAL SLICE that has $n' = 5n$ vertices such that $G$ has a clique on $n/2$ vertices or more than $k$ edges if and only if $G'$ has a $\left\lceil \frac{5n}{2} \right\rceil$ - clique or more than $\left\lceil e(\left\lceil \frac{5n}{2} \right\rceil)/2 \right\rceil$ edges.
Half Clique Central Slice

Let \( V = \{v_1, \ldots, v_n\} \). Build \( G' \) by adding vertices \( R = \{r_1, \ldots, r_{2n}\} \) and \( S = \{s_1, \ldots, s_{2n}\} \). Represent edges with variables \( \{y_{i,j} \mid 1 \leq i, j \leq 5n\} \).

Fix the \( e(4n) + 4n^2 \) edges \( F \) between vertices within \( R \) and \( S \) and between \( R, S \) and \( V \) as follows.

Set \( y_{i,j} \) so that no edge exists between \( r_i \) and \( s_i \) \( 1 \leq i \leq 2n \), \( R \) is a clique, and each vertex in \( V \) is connected to each vertex in \( R \). This fixes \( 4n^2 + n \) of the edges \( F \) leaving \( 8n^2 - 3n \) edges to be fixed.
Half Clique Central Slice

Let \( r = \left\lceil e\left(\lceil 5n/2 \rceil/2\right) \right\rceil - (4n^2 + n) \). Clearly, \( r \leq 8n^2 - 3n \). Fix the remaining vertices so that \( F \) has \( r-p \) edges.

If \( G \) has an \( (n/2) \)-clique, \( G' \) has a \( (5n/2) \)-clique. On the other hand, if \( G' \) has a \( (5n/2) \)-clique, \( G \) must have a \( (n/2) \)-clique because no clique of this size can include \( S \).

Also, \( G \) has more than \( p \) edges iff \( G' \) has more than \( \left\lceil e\left(\lceil 5n/2 \rceil/2\right) \right\rceil - p \) edges.
Half Clique Central Slice

- The membership of a graph G in HALF-CLIQUE can be determined by determining membership of G’ in HALF-CLIQUE CENTRAL SLICE. This the latter problem is \textbf{NP}-hard. Because it is in \textbf{NP}, it is \textbf{NP}-complete.