Indirect Storage Access Function

Let $K = 2^k$ and $L = 2^l$. Consider the indirect storage access function $f_{ISA}^{(k,l)}(a, x_{K-1}, ... , x_0, y) = y_b$, $b = |x_a|_i$. $a$ is a $k$-bit address vector, $x_j$ is an $l$-bit address vector, and $y$ is an L-bit data vector.

Problem 9.24 gives formula for $f_{mux}(n)$ of size $3 \times 2^n - 2$ that uses $2(2^n - 1)$ instances of address variables.

Applying this to the above formula for $f_{ISA}^{(k,l)}$ gives a formula that has size $O(n^2 / \log n)$.

Neciporuk’s Formula Size Lower Bound

Neciporuk’s lower bound method provides a lower bound to formula size of the same order as the upper bound for this function.

Given $f : B^n \to B$, partition its $n$ variables $X$ into $p$ disjoint sets $X_1, X_2, ... , X_p$.

Let $r_j(f)$ be the no. of different subfunctions of $f$ over $X_j$ when fixing the values of variables in $X - X_j$.

We derive a lower bound on the formula size of $f$ in terms of $r_i(f)$ for $1 \leq i \leq p$. ($r_i(f)$’s depend on the partition used. Choose the partition wisely!)

Theorem For every complete basis $\Omega$ there exists constant $c_{\Omega} = 1/(d+2)$ such that for every $f : B^n \to B$ its formula size satisfies

$$L_{\Omega}(f) \geq c_{\Omega} \sum_{j=1}^{p} \log_2 r_j(f)$$

Proof Let $T$ be a minimal fan-out 1 circuit for $f$ (a tree). Let $n_j$ be number of instances of vars in $X_j$ used in $T$. $L(f) = n_1 + n_2 + ... + n_p$.

Let $T_j$ consist of paths from variables in $X_j$ to root of $T$. Vertices with only one path entering them are called controller vertices. Others are combiner vertices. Assigning values to variables $X - X_p$ each controller computes one of 4 functions on a variable $x$, which we represent by $(a \text{ AND } x) \text{ XOR } b$ for constants $a$ & $b$. (If $a=0$, its $b$, if $a=1$, its $x \text{ XOR } b$.)

Clearly each chain of controllers can be compressed to one controller.
Neciporuk’s Formula Size Lower Bound

Each combiner has at least two inputs on paths from variables in \( X_j \). Thus, if a gate (vertex) has maximum fan-in \( d \), at most \( d-2 \) combiner inputs are constants determined by the values of variables in \( X - X_j \).

By earlier lemma since \( T_j \) on \( X_j \) has \( n_j \) leaves, its number of vertices with fan-in \( \geq 2 \) (combiners) is at most \( n_j - 1 \). Also, \( T_j \) has \( \leq 2(n_j - 1) \) edges. Since \( T_j \) has at most one controller per edge plus one at output, it has \( \leq 2n_j - 1 \) controllers.

The number of functions computed by a combiner is at most \( 2^{d-2} \) because it has at most \( d-2 \) constant inputs. At most 4 functions are computed by a controller.

Let \( m = n_j \). Thus, the max number of subfunctions computed by \( T_j \) is at most \( 2^{(d-2)(m-1)} + 4^{(2m-1)} \) \( \leq 2^{(d+2)m} \). Thus, \( (d+2)n_j \geq (\log_2 r_j(f)) \) and the theorem follows. ♥

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Formula Size Lower Bound

**Lemma** Let \( L = 2^l = n \) and \( k = \lceil \log_2 (n/l) \rceil \). Then, \( L_{-\Omega}(f_{ISA}^{(k,l)}) = \Omega(n^2 / \log n) \)

**Proof** Recall that

\[
f_{ISA}^{(k,l)}(a, x_{K-1}, \ldots, x_0, y) = y_b, \ b = |x_0|.
\]

where \( a \) is a \( k \)-bit address vector, \( x_j \) is an \( l \)-bit address vector, and \( y \) is an \( L \)-bit data vector.

Let \( p = K = 2^k \) and let \( X_j \) contain the variables in \( x_j \) and possibly other variables that are fixed (which cannot increase \( r_j(f) \)).

If \( |a| = j \), \( r_j(f) \geq 2^L \) because each of \( 2^L \) assignments of \( y \) defines a different function of the variables in \( X_j \).

It follows that the formula size is at least \( cL_{-\Omega}KL \). But \( K=2^k \geq n/l \) where \( l = \log_2 n \) and \( L=n \), from which the result follows. ♥