# 8

# Spine and Radial Drawings

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## 8.1 Introduction

A *layered drawing* of a graph is a drawing such that the vertices are constrained to lie on geometric layers that can be lines, circles, or other kinds of curves. Partitioning the vertices into distinct layers can be an effective way to emphasize some structural properties of the graph; in many cases this is required in some real-world applications to convey the so-called *semantic constraints* [Sug02].

In this chapter, we concentrate on layered drawings of undirected graphs, where the edges are not constrained to be monotone in a given direction. Conversely, this is typically a basic requirement in the layered drawings of directed graphs or hierarchies, where all edges must flow in a common direction (usually the vertical one), according to their orientation. Layered drawings algorithms for directed graphs are extensively investigated in Chapter 13.

Although it is theoretically interesting to study layered drawings where the layers can be curves of any type, it is rather difficult to extract properties and design algorithms if the layers do not have a quite "regular" shape. Indeed, most of the literature assumes that the layers are either parallel straight lines or concentric circles, which is also the most common requirement in real-world application domains. Therefore, we only give an overview of the results on layered drawings where layers are straight lines or circles. We call the first family of drawings *spine drawings* and the second family *radial drawings*.

The remainder of this chapter is structured as follows. We first give formal definitions that are needed in the chapter and describe a unified investigation framework for spine and radial drawings (Section 8.2). Then, we investigate the results on spine and radial drawings in a general scenario (Section 8.3); this scenario has the only requirement that the vertices are placed on layers. Results on scenarios that consider additional constraints are investigated in Section 8.4. Finally, we mention some topological and geometric problems related to the spine and radial drawability of a graph (Section 8.5) and we give conclusions (Section 8.6).

## 8.2 A Unified Framework for Spine and Radial Drawings

### 8.2.1 Definitions

A *drawing* $\Gamma$ of a graph $G$ is a geometric representation of $G$ such that each vertex $u$ of $G$ is mapped to a distinct point $p_u$ of the plane and each edge $(u, v)$ of $G$ is drawn as a simple Jordan curve with end-points $p_u$ and $p_v$. Drawing $\Gamma$ is *planar* if two distinct edges never intersect except at common end-vertices. $G$ is *planar* if it admits a planar drawing. A planar drawing $\Gamma$ of $G$ partitions the plane into topologically connected regions called the *faces*. The unbounded face is called the *external face* and the other faces are called *internal faces*. The *boundary* of a face is its delimiting circuit (not necessarily a simple cycle) described by the circular list of its edges and vertices. The *boundary* of the external face, also called the *external boundary*, is the circular list of edges and vertices delimiting the unbounded region. If the graph is biconnected, the boundary of each face is a simple cycle. An *embedding* of a planar graph $G$ is an equivalence class of planar drawings that determine the same set of faces, i.e., the same set of face boundaries. A planar graph $G$ with a given embedding is called an *embedded planar graph*. In this chapter, we only deal with planar graphs and planar drawings. From a practical point of view, if a graph is not planar, one can think of applying a planarization algorithm on it in order to find a planar embedding with dummy vertices that replace crossings [DETT99].

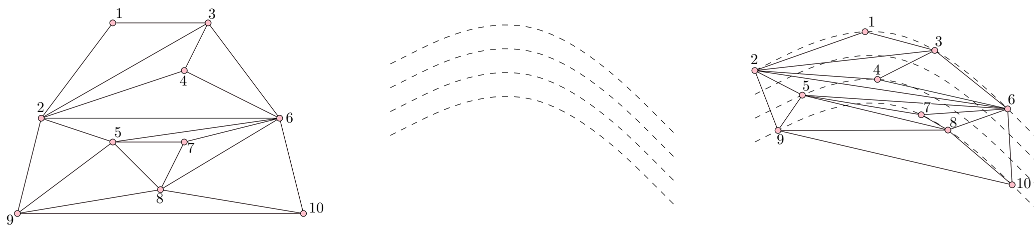
A drawing $\Gamma$ of $G$ such that the edges are represented as a polygonal chain is a *polyline drawing*. A *bend* along an edge $e$ of $\Gamma$ is a common point between two consecutive straight-line segments that form $e$. If every edge of $\Gamma$ has at most $b$ bends, $\Gamma$ is a *$b$-bend drawing* of $G$. A 0-bend drawing is also called a *straight-line drawing*.

Let $\gamma_1$ and $\gamma_2$ be two curves. Curves $\gamma_1$ and $\gamma_2$ are *parallel* if every normal to one curve is a normal to the other curve and the distance between the points where the normals cut the two curves is a constant. Examples of parallel curves are parallel straight lines or concentric circles. A *set of layers* is a set of pairwise parallel curves; each curve in the set is called a *layer*. Given a set of layers it is possible to order the layers according to the order they are encountered while walking along a straight line normal to all of them. More precisely, let $\mathcal{C}$ be a set of layers, and let $l_n$ be a normal to all the layers in $\mathcal{C}$. Let $p_i$ be the intersection point between $l_n$ and $\gamma_i \in \mathcal{C}$ and let $p_j$ be the intersection point between $l_n$ and $\gamma_j \in \mathcal{C}$. Given an orientation for $l_n$, we have that $\gamma_i$ is before $\gamma_j$ if $p_i$ is encountered before $p_j$ while walking along $l_n$ according to the given orientation, $\gamma_i$ is after $\gamma_j$ otherwise. In the following, given a set of layers denoted as $\gamma_0, \dots, \gamma_{k-1}$, we always assume that $\gamma_i$ is before $\gamma_{i+1}$ for each $0 \leq i \leq k-1$.

**DEFINITION 8.1** Let $G = (V, E)$ be a planar graph, and let $\mathcal{C} = \{\gamma_0, \dots, \gamma_{k-1}\}$ be a set of layers, with $k \leq n$. A *$k$-layered drawing* of $G$ on $\mathcal{C}$ is a polyline planar drawing $\Gamma$ of $G$ such that each vertex $v \in V$ is represented in $\Gamma$ as a point $p_v \in \gamma_i$ $(0 \leq i \leq k-1)$.

An example of a 4-layered drawing is shown in Figure 8.1. A $k$-layered drawing will be simply called a *layered drawing* when we are not interested in the number of layers.

Let $\Gamma$ be a $k$-layered drawing of a graph $G$, and let $e = (u, v)$ be an edge of $G$ such that $u$ is drawn in $\Gamma$ on layer $\gamma_i$ and $v$ is drawn in $\Gamma$ on layer $\gamma_j$ $(0 \leq i, j \leq k-1)$. The *span* of $e$



(a) (b) (c)

**Figure 8.1** (a) A planar graph $G$. (b) A set $\mathcal{C}$ of four layers. (c) A 4-layer, 0-bend drawing of $G$ on $\mathcal{C}$

in $\Gamma$ is $|i-j|$. An *intra-layer edge* is an edge with span equal to 0, i.e., an edge connecting vertices that are on the same layer. A *long edge* is an edge with span greater than 1.

In the following, we shall consider two special cases of layered drawings: *spine drawings*[1] and *radial drawings*, which are defined as follows.

**DEFINITION 8.2** A *k-spine drawing* of a planar graph is a planar $k$-layered drawing such that the layers are horizontal straight lines, called *spines*.

**DEFINITION 8.3** A *k-radial drawing* of a planar graph is a planar $k$-layered drawing such that the layers are concentric circles.

For a $k$-spine drawing, we denote the set of layers as $\mathcal{C} = \{L_0, \ldots, L_{k-1}\}$ and we assume that they are ordered from the highest to the lowest, i.e., $L_0$ is the topmost line and $L_{k-1}$ is the bottommost one. For a $k$-radial drawing, we denote the set of layers as $\mathcal{C} = \{C_0, \ldots, C_{k-1}\}$, and we assume that they are ordered from the more external to the innermost, i.e., $C_0$ is the circle with the largest radius and $C_{k-1}$ is the one with the smallest radius. When we are not interested in distinguishing between spine and radial drawings, we will generically denote the layers as $\mathcal{C} = \{\gamma_0, \ldots, \gamma_{k-1}\}$. If a planar graph $G$ admits a $k$-spine, $b$-bend drawing ($k$-radial, $b$-bend drawing), we say that $G$ is *k-spine, b-bend drawable* (*k-radial, b-bend drawable*).

We conclude this section with some definitions about Hamiltonicity that will be used in the following. A *Hamiltonian cycle* of $G$ is a simple cycle that contains all vertices of $G$. A graph $G$ that admits a Hamiltonian cycle is said to be *Hamiltonian*. A planar graph $G$ is *sub-Hamiltonian* if either $G$ is Hamiltonian or $G$ can be augmented with dummy edges (but not with dummy vertices) to a graph that is Hamiltonian and planar. We denote by $\mathsf{aug}(G)$ a planar Hamiltonian graph obtained by $G$ by possibly adding edges (if $G$ is Hamiltonian then $\mathsf{aug}(G) = G$). A *subdivision* of a graph $G$ is a graph obtained from $G$ by

[1]Drawings on a set of horizontal layers are often called *layered drawings* in the literature. Since in this chapter we use the term *layered drawing* to denote the more general case of a drawing on any set of parallel curves, we use the term *spine drawings* when the layers are straight lines. This term is taken from the theory of book embeddings, which can be regarded as drawings on a single horizontal line, usually called the *spine* of the book embedding.

replacing each edge by a path with at least one edge. Internal vertices on such a path are called *division vertices*. It is easy to see that any planar graph always admits a subdivision that is sub-Hamiltonian. Let $G$ be a planar graph and let $\mathsf{sub}(G)$ be a sub-Hamiltonian subdivision of $G$ (if $G$ is sub-Hamiltonian, then $\mathsf{sub}(G) = G$). The graph $\mathsf{aug}(\mathsf{sub}(G))$ is called a *Hamiltonian augmentation of* $G$ and will be denoted as $\mathsf{Ham}(G)$ (if $G$ is Hamiltonian then $\mathsf{Ham}(G) = G$). A Hamiltonian cycle of $\mathsf{Ham}(G)$ is called an *augmenting Hamitonian cycle* of $G$.

### 8.2.2 Scenarios

In the following, we are interested in characterizing $k$-spine, $b$-bend drawable and $k$-radial, $b$-bend drawable graphs for different values of $k$ and $b$. We are also interested in the drawability testing problems, i.e., in studying the complexity of deciding whether a given planar graph is $k$-spine, $b$-bend drawable ($k$-radial, $b$-bend drawable). More precisely, we consider the following two problems.

**Characterization Problem.** Let $k$ and $b$ be two given integers. What is the largest class of $k$-spine, $b$-bend drawable ($k$-radial, $b$-bend drawable) graphs?

**Drawability Testing Problem.** Let $k$ and $b$ be two given integers and let $G$ be a planar graph. What is the complexity of deciding whether $G$ is $k$-spine, $b$-bend drawable ($k$-radial, $b$-bend drawable)?

The study of these two problems is motivated by the fact that, for aesthetic reasons, one can be interested in keeping the number of layers and the number of edge bends in a layered drawing as small as possible. Observe that every planar graph $G$ with $n$ vertices is $k$-spine 0-bend drawable, for some value of $k \leq n$. Indeed, it is known that $G$ admits a planar straight-line drawing $\Gamma$ [Fár48], and at most $n$ distinct horizontal parallel layers are sufficient to intersect all vertex-points in $\Gamma$. Furthermore, since $G$ also admits a planar straight-line drawing on an integer grid of size $O(n) \times O(n)$ [dPP90], $G$ is always $k$-spine 0-bend drawable within an $O(n^2)$ area. With analogous considerations, every planar graph with $n$ vertices is $k$-radial 0-bend drawable for some value of $k \leq n$.

The Characterization Problem and the Drawability Testing Problem can be studied within different scenarios, depending on the additional constraints that one can define. We first consider the two problems without any additional constraint. We will refer to this scenario as the *general scenario*. We then consider the same problems with some of the following additional constraints:

**Intra-layer edges not allowed.** Many results in the literature assume that there is no intra-layer edge in a layered drawing. For example, avoiding intra-layer edges in a $k$-layered drawing could be important to put in evidence a $k$-partite structure of the graph. Indeed, a $k$-layered drawing of a graph $G = (V, E)$ implicitly defines a partition of the set $V$ into $k$ sets $V_0, V_1, \ldots, V_{k-1}$, where each set $V_i$ is the set of vertices drawn on layer $\gamma_i$. Layered drawings with no intra-layer edges will be called *upright drawings*.

**Assigned vertex partitioning.** In some cases, the partitioning of the vertices can be given as a part of the input. In these cases, the vertex partition determined by the layered drawing has to preserve the one given in the input. Layered drawings where the partition of the vertices is given will be called *partitioned layered drawings*.

**Long edges not allowed.** Edges that span more than one level are more difficult to follow by the human eye than edges connecting vertices on consecutive layers.

Thus another common constraint in a layered drawing is to avoid long edges. Layered drawings with no long edges will be called *proper drawings*.

**Assigned layers.** In the general scenario, we are assuming that only the number and the type (spines or circles) of layers are given. However, one can consider the case when also the distance between every two consecutive layers is given as part of the input. Having the distances of the layers assigned as a part of the input may change the answer to both the Characterization and the Drawability Testing Problem.

## 8.3 Results in the General Scenario

### 8.3.1 Spine Drawings in the General Scenario

We start by considering the easiest case for $k$-spine drawings, i.e., the case when $k = 1$. A trivial result is that if only 0 bends per edge are allowed we can only draw forests of paths, and therefore, the drawability test can be executed in $O(n)$ time, where $n$ is the number of vertices of the input graph.

**PROPOSITION 8.1** A planar graph is 1-spine, 0-bend drawable if and only if it is a forest of paths.

If one bend per edge is allowed the problem of computing a 1-spine, 1-bend drawing of a planar graph $G$ is equivalent to that of computing a book embedding of $G$ on two pages. A *book embedding* of a graph $G = (V, E)$ consists of a total order $<_\sigma$ of $V$ and a partition of $E$ into $p$ sets, called *pages*, such that there are no two edges $(u, v)$ and $(w, z)$ in the same page with $u <_\sigma w <_\sigma v <_\sigma z$. The *pagenumber* of a graph $G$ is the minimum value $p$ for which $G$ admits a book embedding with $p$ pages.

A book embedding can be seen as a drawing of $G$ where: (i) all vertices are drawn along a straight line, called the *spine*, according to the total order $<_\sigma$, (ii) each edge is assigned to one among $p$ half-planes having the spine as a common boundary, (iii) no two edges in the same page cross (see Figure 8.2). It is not difficult to prove that if two edges can be drawn without crossings on a half-plane with the endvertices on the boundary of the half-plane, then they can be drawn without crossings as two polylines with one bend on the same half-plane and with the end-vertices in the same position (see also Figure 8.2). Since a straight line on a plane define two half-planes we have the following lemma.

**LEMMA 8.1** A planar graph is 1-spine, 1-bend drawable if and only if it has pagenumber two.

Bernhart and Kainen [BK79] prove that a graph has pagenumber at most two if and only if it is sub-Hamiltonian. If a graph $G$ admits a book embedding on two pages, then let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of $G$ ordered according to the total ordering $<_\sigma$ of the book embedding. An augmenting Hamiltonian cycle of $G$ is $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-2}, v_{n-1})$, $(v_{n-1}, v_0)$ where each edge $(v_i, v_{i+1})$ is either an edge of $G$ or a dummy edge that can be added to $G$ without violating planarity (see Figure 8.3 for an example). Conversely, if $G$ is sub-Hamiltonian there exists an augmenting Hamiltonian cycle $H$ (possibly obtained by adding some edges) in $G$. Choose an embedding $\Psi$ of $G$ with an edge $e$ of $H$ on the external face. By removing $e$ we have a path $P$ containing all the vertices of $G$. Define the total

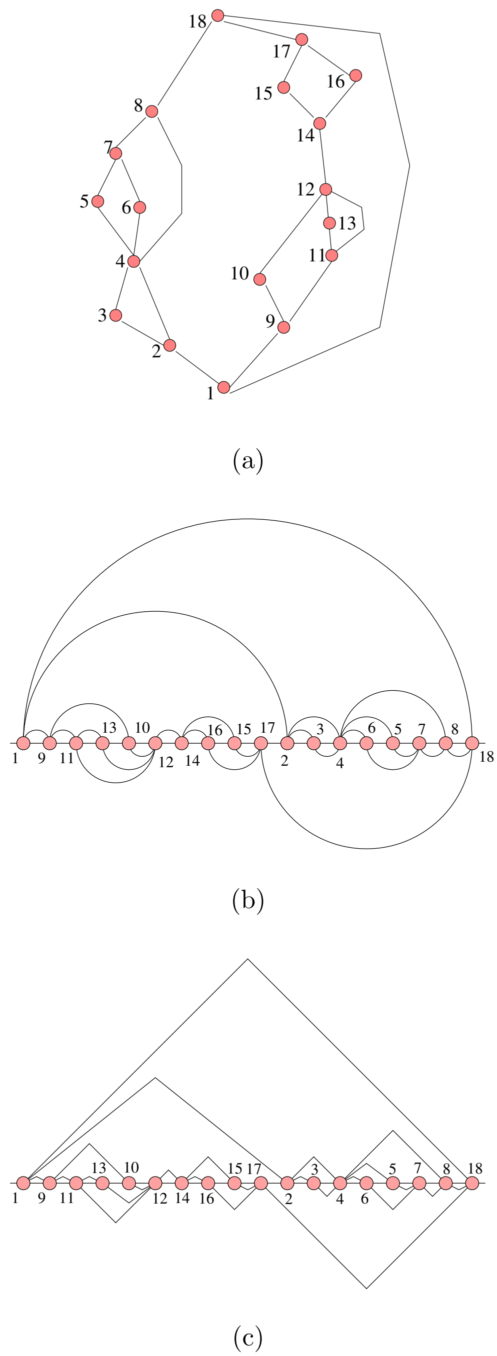



**Figure 8.2** (a) A planar graph $G$. (b) A book embedding of $G$ on two pages: the total order of the vertices is the left-to-right order of the vertices along the horizontal line, while the two pages are represented by the two half-planes defined by the same line. (c) A 1-spine, 1-bend drawing of $G$.

order $<_\sigma$ according to the order the vertices of $G$ are encountered while walking along $P$. The edges in $P$ can be assigned to one of the two pages. Edge $e$ can also be assigned to the same page as the edges in $P$. All the remaining edges of $G$ are either inside or outside $H$ in the embedding $\Psi$. Those that are inside $H$ are assigned to the same page as all the edges of $H$, those that are outside are assigned to the other page. There cannot be two edges $(u, v)$ and $(w, z)$ in the same page such that $u <_\sigma w <_\sigma v <_\sigma z$, because otherwise there would be a crossing in the embedding $\Psi$.

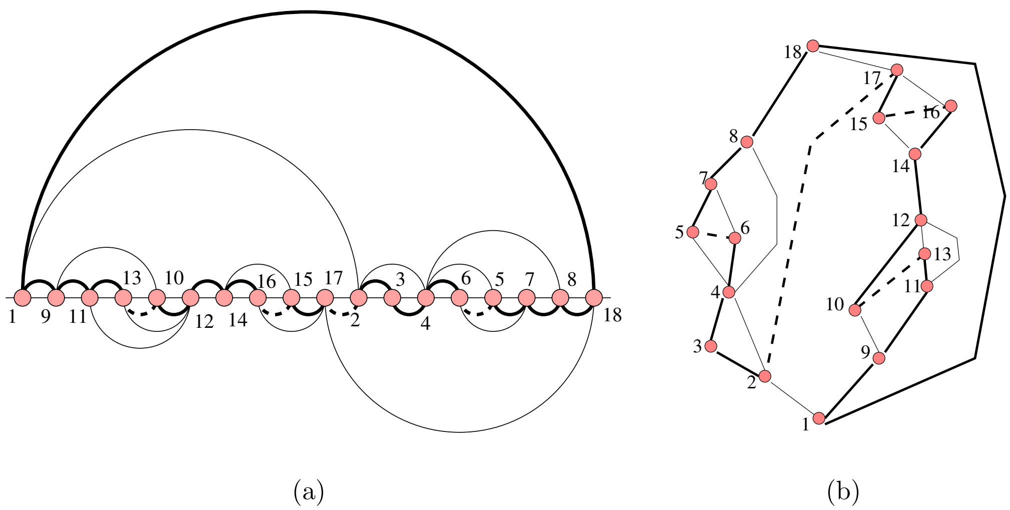



**Figure 8.3** An augmentation of the planar graph of Figure 8.2 to a planar Hamiltonian graph.

Based on the result of Bernhart and Kainen and on Lemma 8.1, we have that the class of graphs that admit a planar 1-spine, 1-bend drawing is the class of sub-Hamiltonian graphs. Since testing sub-Hamiltonicity is $\mathcal{NP}$-complete [Wig82], we have that testing a graph for 1-spine, 1-bend drawability is $\mathcal{NP}$-complete, too.

**Theorem 8.1** *A planar graph is* 1*-spine,* 1*-bend drawable if and only if it is sub-Hamiltonian.*

Although Theorem 8.1 gives a complete characterization of 1-spine, 1-bend drawable graphs, such graphs cannot be recognized efficiently; thus it is worth investigating some specific families of graphs that are sub-classes of the sub-Hamiltonian graphs and that can be recognized efficiently. Among them we recall here: outerplanar graphs [BK79] (that coincide with the graphs having pagenumber one), series-parallel graphs [DDLW06, RM95], planar bipartite graphs [ddMP95], square grids, and $X$-trees [CLR87].

If two bends per edge are allowed, then every planar graph is drawable on one spine. This result is a consequence of a result by Kaufmann and Wiese [KW02] about point-set embeddability. Given a planar graph $G = (V, E)$ and a set $S$ of points in the plane such that $|S| = |V| = n$, a *point-set embedding* of $G$ onto $S$ is a planar drawing of $G$ such that each vertex of $G$ is represented as a point of $S$. Kaufmann and Wiese [KW02] prove that

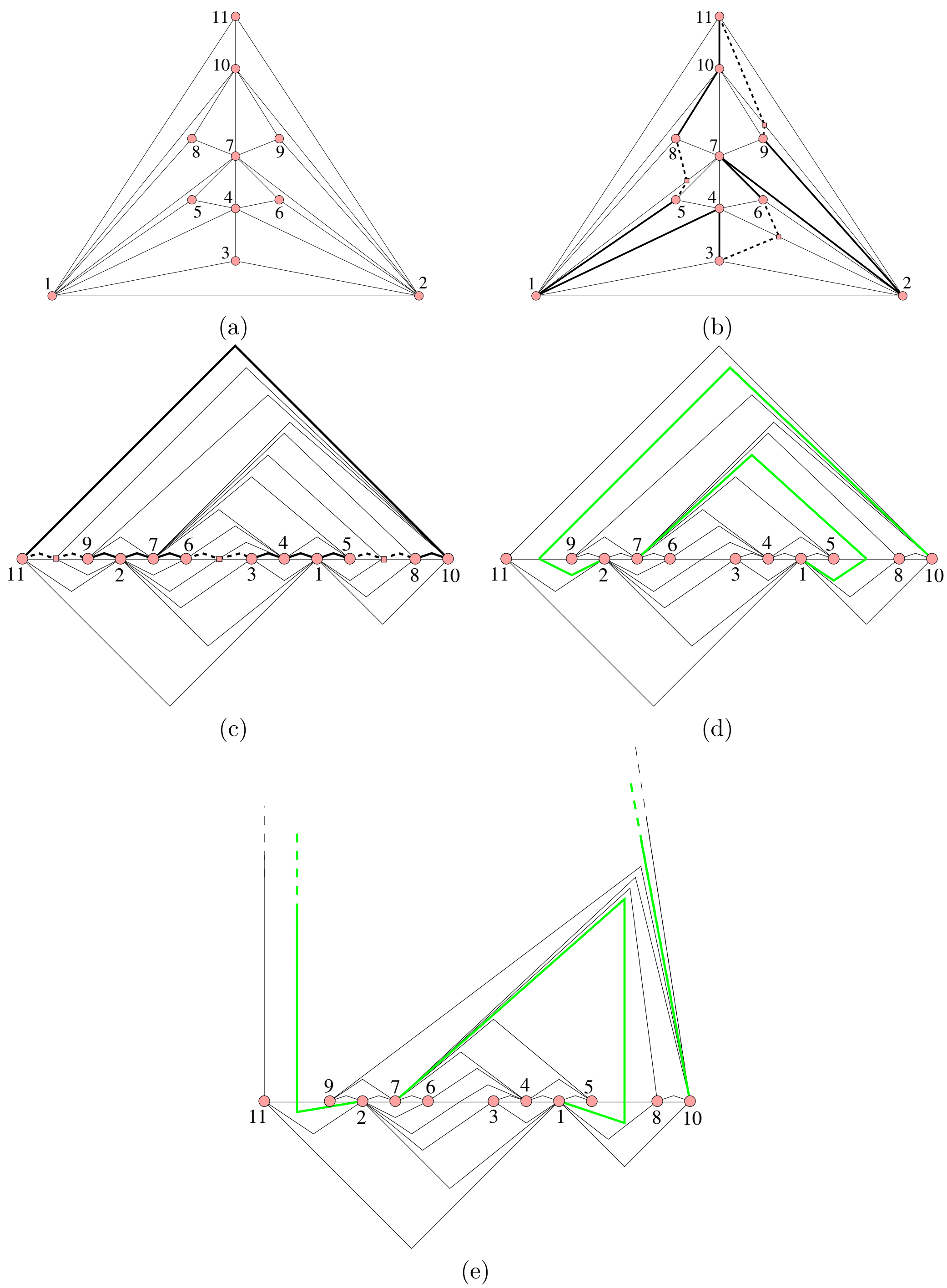



**Figure 8.4** (a) A non-Hamiltonian graph $G$. (b) A Hamiltonian augmentation $\mathsf{Ham}(G)$ of $G$. (c) A 1-spine, 1-bend drawing of $\mathsf{Ham}(G)$. (d) A 1-spine, 3-bend drawing of $G$; the edges with 3 bends are highlighted. (e) A 1-spine, 2-bend drawing of $G$ obtained by rotating the segments of the edges that had 3 bends in the previous picture.

every planar graph $G$ admits a point-set embedding on any given set of points such that every edge of $G$ is represented as a polyline with at most 2 bends. Such a drawing can be computed in $O(n \log n)$ time. In order to compute a planar 1-spine, 2-bend drawing of a planar graph $G$, it is sufficient to choose a set of $n$ collinear points and then apply the Kaufmann and Wiese algorithm. As a consequence, the following theorem holds.

**Theorem 8.2** *Every planar graph is* 1*-spine,* 2*-bend drawable.*

Although the paper by Kaufmann and Wiese is about point-set embeddings and does not mention book embeddings, their drawing technique can be regarded as an extension of the technique used to compute a 1-spine, 1-bend drawing of a Hamiltonian graph. Kaufmann and Wiese compute a Hamiltonian augmentation $\mathsf{Ham}(G)$ of the input graph $G$ such that each edge of $G$ is subdivided at most once (see Figure 8.4). Since $\mathsf{Ham}(G)$ is Hamiltonian it admits a 1-spine, 1-bend drawing by Theorem 8.1. An edge $e = (u, v)$ that has been subdivided by a division vertex $w$, is represented in the 1-spine, 1-bend drawing of $\mathsf{Ham}(G)$ by two edges $(u, w)$ and $(w, v)$ each one drawn with at most one bend. Removing the division vertex $w$ we obtain another bend on edge $e$ at the point $p_w$ where $w$ was drawn. This removal would give rise to at most three bends per edge (see Figure 8.4). However, it is possible to remove this third bend by suitably rotating the segments incident to $p_w$ (see Figure 8.4). The drawing technique described above requires to compute a Hamiltonian augmentation $\mathsf{Ham}(G)$ of $G$. Kaufmann and Wiese describe a Hamiltonian augmentation technique that runs in $O(n)$ time and subdivides each edge at most once. Details about different Hamiltonian augmentation techniques are given in Section 8.5.1.

We conclude this discussion about 1-spine, 2-bend drawings by further remarking the connection between them and book embeddings. The 1-spine, 2-bend drawing of the input graph $G$ is obtained from a 1-spine, 1-bend drawing of the Hamiltonian graph $\mathsf{Ham}(G)$. An edge $e = (u, v)$ that has been subdivided by a division vertex $w$, is represented in the 1-spine, 1-bend drawing of $\mathsf{Ham}(G)$ by two edges $(u, w)$ and $(w, v)$ that may be on the two different half-planes defined by the spine. This means that a 1-spine, 2-bend drawing of a planar graph $G$ can be seen as a book embedding of $G$ on two pages, where each edge is not required to be on one page only but is allowed to cross the spine at most once. A book embedding where edges are allowed to cross the spine is also called a *topological book embedding*. Therefore, Theorem 8.2 implies that every planar graph has a topological book embedding on two pages where each edge crosses the spine at most once. Since two bends are sufficient to draw all planar graphs on a single spine, it does not make sense to further investigate 1-spine, $b$-bend drawings for $b > 2$.

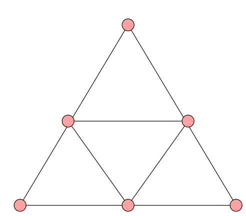
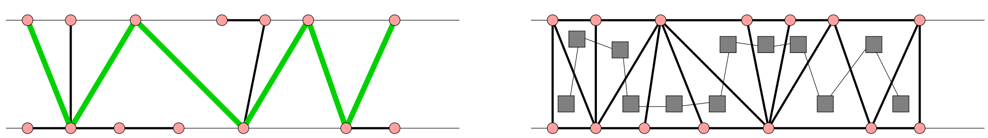


**Figure 8.5** An outerplanar graph that does not admit a 2-spine, 0-bend drawing.

Consider now the case when two spines are given. It is immediate to see that if a graph admits a 2-spine, 0-bend drawing, then it is outerplanar (i.e., it admits a planar embedding such that all the vertices are on the external face). Indeed, in a 2-spine, 0-bend drawing each vertex is either a topmost vertex or a bottommost vertex, and therefore, since the edges are straight lines, it is on the external face. Observe however that not all outerplanar graphs admits a 2-spine, 0-bend drawing. The graph in Figure 8.5 is the smallest (in terms of number of vertices) outerplanar graph that is not 2-spine, 0-bend drawable.

Some preliminary results about 2-spine, 0-bend drawability were presented by Felsner et al. [FLW03], who characterize trees that are 2-spine, 0-bend drawable. They prove that a tree $T$ admits a 2-spine, 0-bend drawing if and only if there exists a path $P$ in $T$ such that removing $P$ from $T$ we are left with a collection of vertex disjoint paths (see Figure 8.6). A characterization of (outer)planar graphs that admit a planar 2-spine, 0-bend drawing has been given by Cornelsen et al. [CSW04]. They first consider biconnected outerplanar graphs and prove that a biconnected outerplanar graph $G$ admits a 2-spine, 0-bend drawing if and only if its internal faces induce a path in the dual graph of $G$ (see Figure 8.6). The *dual graph* $G^*$ of a planar graph $G$ is a multigraph that has a vertex for each face of $G$ and an edge between two vertices $f$ and $g$ if the two faces represented by $f$ and $g$ share an edge. For general simply connected outerplanar graphs Cornelsen et al. [CSW04] describe a decomposition of an outerplanar graph $G$ into components like paths, trees and biconnected outerplanar components and describe necessary and sufficient conditions that these components must satisfy for the 2-spine, 0-bend drawability of $G$. Therefore, the outerplanar graphs whose components satisfy these conditions are exactly the planar graphs that are 2-spine, 0-bend drawable. The necessary and sufficient conditions described in [CSW04] cannot be shortly summarized. Intuitively, they guarantee that each single component is 2-spine, 0-bend drawable and that the vertices shared by different components are drawn so that the drawings of the different components can be merged together. Finally, Cornelsen et al. [CSW04] prove that the necessary and sufficient condition above can be tested in $O(n)$ time, thus proving that 2-spine, 0-bend drawability can be tested in linear time.

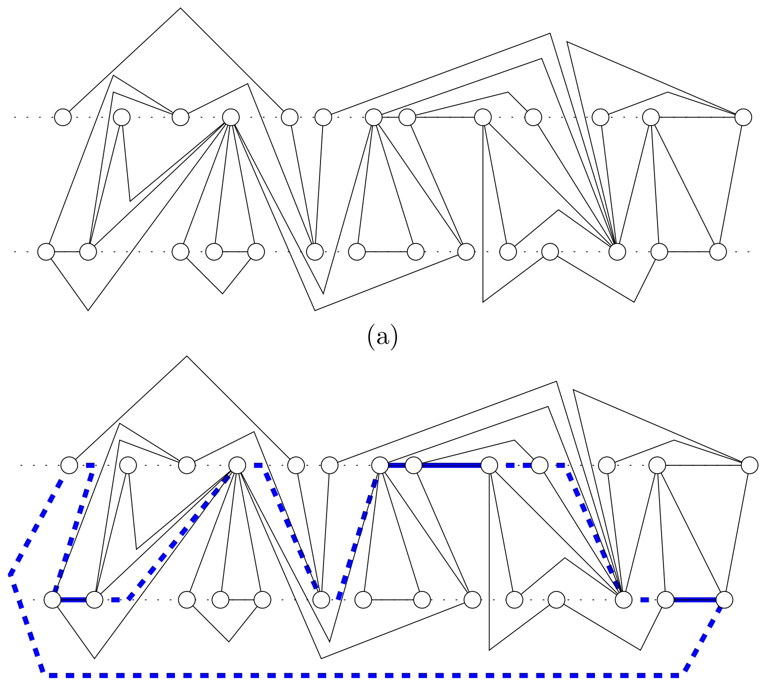


(a) (b)

**Figure 8.6** (a) A 2-spine, 0-bend drawable tree. The removal of the highlighted path leaves a set of paths. (b) A 2-spine, 0-bend drawing of a biconnected outerplanar graph $G$ such that the inner faces of $G$ induce a path in the dual graph of $G$. In the picture, the node of the dual graph corresponding to the outer face is not shown.

Drawings on two spines with at most one bend per edge are the subject of [DDLS06] where, in fact, $k$-spine, 1-bend drawings have been studied. In [DDLS06], a $k$-spine, 1-bend drawing is considered as an extension of a 2-page book embedding where the spines are more than one, and it is proved that, for any fixed $k \geq 2$, not all planar graphs are $k$-spine, 1-bend drawable. The proof is based on the observation that, if a graph admits a $k$-spine, 1-bend drawing, it must exist a special cycle, called *cutting cycle* (see Figure 8.7), removing which we are left with $(k-1)$-spine, 1-bend drawable subgraphs. The cutting cycle is actually a sequence of vertices that may or may not correspond to an actual cycle in the graph. Instead, the sequence of vertices is such that, if dummy edges are inserted between non-adjacent vertices that are consecutive in the sequence, then the resulting drawing is still a $k$-spine, 1-bend drawing. The following lemma holds.

**LEMMA 8.2** If $G$ is a maximal planar graph that is $k$-spine, 1-bend drawable for $k \geq 2$, then there exists a simple cycle $C$ in $G$ such that $G \setminus C$ is $(k-1)$-spine, 1-bend drawable.



(a)

(b)

**Figure 8.7** (a) A planar 2-spine, 1-bend drawing of a planar graph $G$. (b) A cutting cycle of $G$. Figure taken from [DDLS06].

We can use now the necessary condition expressed by Lemma 8.2 to construct, for any fixed $k \geq 1$, a maximal planar graph $N^k$ that is not $k$-spine, 1-bend drawable. Graph $N^1$ is the graph shown in Figure 8.8 (it is the same graph as in Figure 8.4), which is a non-Hamiltonian graph. Graph $N^k$ is obtained from $N^1$ by replacing each black vertex with a copy of $N^{k-1}$ and triangulating the result (see Figure 8.8).

The proof that $N^k$ is not $k$-spine, 1-bend drawable is by induction on $k$. $N^1$ is not 1-spine, 1-bend drawable by Theorem 8.1 because it is not Hamiltonian. Let $N^{k-1}$ be not $(k-1)$-spine, 1-bend drawable and assume by contradiction that $N^k$ is $k$-spine, 1-bend drawable. By Lemma 8.2 there exists a simple cycle $C$ in $N^k$ whose removal leaves us with $(k-1)$-spine, 1-bend drawable subgraphs. Since each copy of $N^{k-1}$ is not $(k-1)$-spine, 1-bend drawable, then $C$ contains at least one vertex for each copy of $N^{k-1}$. Also, since each copy of $N^{k-1}$ is inside a triangle of white vertices we have that also all white vertices must be in $C$. However, this would imply that $N^1$ is Hamiltonian.

**Theorem 8.3** *For each integer $k \geq 1$, there exists a planar graph that is not $k$-spine, 1-bend drawable.*

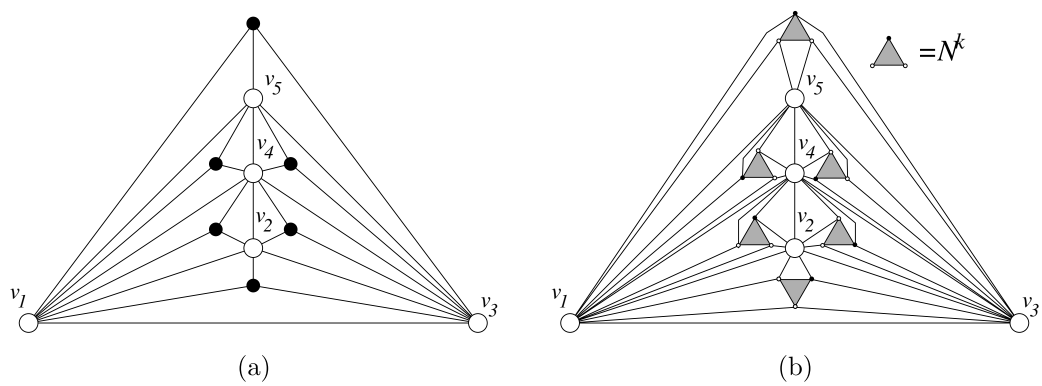



**Figure 8.8** (a) A graph that is not 1-spine, 1-bend drawable. (b) A graph that is not $k$-spine, 1-bend drawable. Figure taken from [DDLS06].

Motivated by the fact that not all planar graphs are $k$-spine, 1-bend drawable, in [DDLS06] the complexity of deciding whether a planar graph is $k$-spine, 1-bend drawable is studied, and it is proved that this problem is $\mathcal{NP}$-complete. The reduction is from the MAXIMAL PLANAR EXTERNAL HAMILTONIAN CIRCUIT problem, i.e., the problem of deciding whether a planar embedded graph contains a Hamiltonian circuit with an edge on the external face. In [DDLS06], a construction is described that, given a maximal planar graph $G$, produces a maximal planar graph $H^k(G)$ that is $k$-spine, 1-bend drawable if and only if $G$ is externally Hamiltonian.

**Theorem 8.4** *The problem of deciding whether a given planar graph is $k$-spine, 1-bend drawable is $\mathcal{NP}$-complete for any fixed $k \geq 1$.*

For the special case of $k = 2$, a complete characterization of 2-spine, 1-bend drawable graphs is given in [DDLS06]. In this case, the necessary condition expressed by Lemma 8.2 can be better detailed. Namely, after removing the cutting cycle, we are left with a set of disjoint paths whose endvertices are adjacent (or can be made adjacent) to the cutting cycle and that satisfy some additional properties (see Figure 8.9). It can be proved that this necessary condition is also sufficient. Graphs whose vertices can be covered by a cycle and a set of vertex-disjoint paths whose end-vertices are connected to the cycle are called *(sub-)Hamiltonian-with-handles graphs* in [DDLS06], which appears as an extension of (sub-)Hamiltonian graphs.

**Theorem 8.5** *A planar graph is 2-spine, 1-bend drawable if and only if it is sub-Hamiltonian-with-handles.*

Theorem 8.4 says that it is $\mathcal{NP}$-complete to recognize sub-Hamiltonian-with-handles graphs. However, there are subclasses of sub-Hamiltonian-with-handles graphs that can be recognized in polynomial time. For example, in [DDLS06] it has been proved that 2-outerplanar graphs are sub-Hamiltonian-with-handles and hence 2-spine, 1-bend drawable.

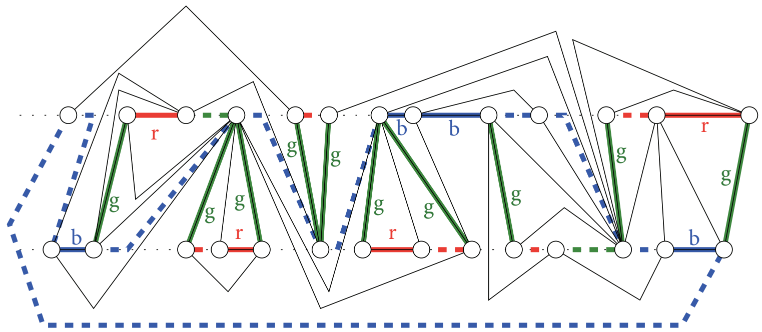



**Figure 8.9** The planar graph of Figure 8.7 covered by a cycle (thick blue edges) and a set of vertex-disjoint paths (thick red edges) whose end-vertices are connected to the cycle (thick green edges). The dashed edges are dummy edges. Labels b, g, and r denote the color of the solid thick edges. The color of the dashed thick edges can be easily inferred.

A complete characterization of the family of $k$-spine, 1-bend drawable graphs is still missing, but Theorem 8.3 tells us that this family is a proper subclass of planar graphs.

A characterization is still missing also for $k$-spine, 0-bend drawable graphs, where some preliminary results have been obtained only for trees. Felsner et al. [FLW03] proved that, for any fixed $k$, it is possible to construct a tree that is not $k$-spine, 0-bend drawable. To produce such a tree, Felsner et al. [FLW03] introduce the notion of *strictness* of a tree T defined as follows. A tree $T$ is 2-strict if it contains a vertex of degree greater than or equal to three. $T$ is $k$-strict if it contains a vertex $v$ adjacent to at least three vertices $u_1$, $u_2$, and $u_3$ such that the subtrees rooted at $u_1$, $u_2$, and $u_3$ are $(k-1)$-strict. In [FLW03] it is proved that a $k$-strict tree is not $(k-1)$-spine, 0-bend drawable. The proof is by induction. A 2-strict tree is not 1-spine, 0-bend drawable since it is not a path. If a tree is $k$-strict, then the three subtrees rooted at $u_1$, $u_2$ and $u_3$ are $(k-1)$-strict and require at least $k$ spine to be drawn. In this case, there is no location for $v$ on the $k$ spines that allows it to connect to the three subtrees without creating a crossing. Based on this result about the strictness of a tree we have that the complete ternary tree of height[2] $k+1$ is not $k$-spine, 0-bend drawable because it is $(k+1)$-strict.

An interesting result shown in [FLW03] is that the strictness of a tree $T$ is closely related to the pathwidth; more precisely, we have that the strictness $s$ of $T$ and the pathwidth $p$ of $T$ are such that $p \leq s \leq p+1$. This implies that if a tree has pathwidth $p$, then it is not $k$-spine, 0-bend drawable for $k < p$. The relationship between the pathwidth $p$ of a tree $T$ and the $k$-spine, 0-bend drawability of $T$ has been further investigated by Suderman [Sud04] who proved that every tree with pathwidth $h$ has a $k$-spine, 0-bend drawing with $p \leq k \leq \lceil 3p/2 \rceil$. Suderman [Sud04] also describes a linear-time drawing algorithm that computes a $k$-spine, 0-bend drawing for a tree with pathwidth $p$, where $k = \lceil 3p/2 \rceil$. In his paper, Suderman studies layered drawings of trees with pathwidth $p$ not only within the general scenario, but also in several different constrained scenarios. We will describe these results in Section 8.4.

A summary of the results described in this section is presented in Table 8.1 (for the Characterization Problem) and in Table 8.2 (for the Drawability Testing Problem).

[2]The *height* is measured as the number of vertices on the path from the root to the deepest leaf.

| | 0 *bends* | 1 *bend* | 2 *bends* |
|---|---|---|---|
| 1 *spine* | paths | sub-Hamiltonian [BK79] | planar [KW02] |
| 2 *spines* | subclass of outerplanar [CSW04] | sub-Hamiltonian-with-handles [DDLS06] | planar [KW02] |
| $k > 2$ *spines* | OPEN (not all trees [FLW03]) | OPEN (not all planar [DDLS06]) | planar [KW02] |

**Table 8.1** Summary of the results about the Characterization Problem for different numbers of spines and bends.

| | 0 *bends* | 1 *bend* | 2 *bends* |
|---|---|---|---|
| 1 *spine* | $O(n)$ | $\mathcal{NP}$-complete [BK79] | always true |
| 2 *spines* | $O(n)$ [CSW04] | $\mathcal{NP}$-complete [DDLS06] | always true |
| $k \geq 2$ *spines* | OPEN | $\mathcal{NP}$-complete [DDLS06] | always true |

**Table 8.2** Summary of the results about the Drawability Testing Problem for different numbers of spines and bends.

### 8.3.2 Radial Drawings in the General Scenario

In this section, we consider $k$-radial, $b$-bend drawings, and start with the case of a single circle. If no bend per edge is allowed, then the class of planar graphs that can be drawn on a circle trivially coincides with the class of outerplanar graphs, which can be recognized in linear time.

**Theorem 8.6** *A planar graph is* 1*-radial,* 0*-bend drawable if and only if it is outerplanar.*

Drawings on one circle and at most one bend per edge have been studied in [DDLW05] where it is proved that every planar graph admits a planar 1-bend drawing on a semi-circle and therefore it is 1-radial, 1-bend drawable. More generally, in [DDLW05] it has been shown that, for every planar graph $G$, it is possible to define a linear ordering $L$ of the vertices of $G$, called *curve embedding*, such that $G$ admits a planar 1-bend drawing on any concave curve $\Lambda$ where the vertices appear along $\Lambda$ in the same order as in $L$. This rather surprising result says that, although not all planar graph can be drawn with one bend on a single spine, it is sufficient to "curve" this spine in order to support all of them. Thus, in one sense, a circle is "more powerful" than any number of spines, because, for any $k > 0$, we know that there are planar graphs that are not $k$-spine, 1-bend drawable.

The algorithm described in [DDLW05] to compute a 1-bend drawing on a semi-circle $\Lambda$ of a maximal planar graph $G$ uses the canonical ordering defined by de Fraysseix, Pach, and Pollack [dPP90]. Let $G$ be a maximal embedded planar graph with external boundary $u, v, w$; a *canonical ordering* of $G$ is an ordering $v_1 = u, v_2 = v, v_3, \ldots, v_{n-1}, v_n = w$ of the vertices of $G$ such that for every $4 \leq k \leq n$:

- the subgraph $G_{k-1}$ of $G$ induced by $v_1, v_2, \ldots, v_{k-1}$ is biconnected and the external boundary $C_{k-1}$ of $G_{k-1}$ contains edge $(u, v)$.

- $v_k$ is on the external face of $G_{k-1}$, and its neighbors in $G_{k-1}$ form a subpath of the path $C_{k-1} - (u, v)$ (see Figure 8.10).

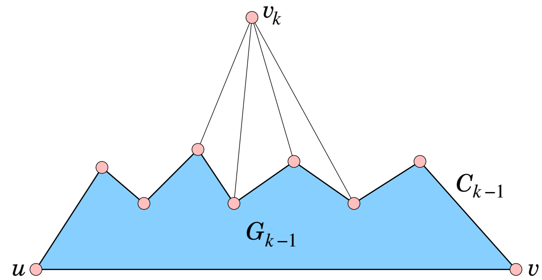



**Figure 8.10** Illustration of the properties of the canonical ordering. Figure taken from [DDLW05].

Once a canonical ordering has been computed, the drawing algorithm in [DDLW05] first draws $G_3$ by placing vertices $v_1$, $v_2$, and $v_3$ at three arbitrary points of $\Lambda$ in the order $v_1$, $v_3$, and $v_2$; the edges between them are drawn as straight-line segments. Vertices $v_4, v_5, \ldots, v_n$ are added one per step. At step $k$ vertex $v_k$ is placed and a planar 1-radial, 1-bend drawing $\Gamma_k$ of $G_k$ is computed. The algorithm guarantees that the following invariants hold for $\Gamma_k$:

- the clockwise order of the vertices along $\Lambda$ is equal to the clockwise order they have on the external boundary $C_k$ of $G_k$;
- each vertex $c$ on the external boundary of $C_k$ is drawn on $\Lambda$ so that there exists two points $\alpha_c$ and $\beta_c$, such that no point of an edge (i.e., no vertex and no internal point of an edge) is encountered going clockwise along $\Lambda$ between $\alpha_c$ and $c$ and between $c$ and $\beta_c$. The arc of $\Lambda$ between $\alpha_c$ and $c$ is called the *left safe region* of $c$ while the arc of $\Lambda$ between $c$ and $\beta_c$ is called the *right safe region* of $c$.

After the drawing of $G_k$ has been computed, vertex $v_{k+1}$ has to be added to the drawing. By the properties of the canonical ordering, $v_{k+1}$ is adjacent to a set of vertices $w_1, w_2, \ldots, w_h$ that are consecutive on the external boundary of $G_k$ and, by the first invariant, are consecutive along $\Lambda$. Vertex $v_{k+1}$ is placed in the right safe region of $w_1$ (i.e., between $w_1$ and $\beta_{w_1}$). By the second invariant, this arc is "free," i.e., it does not contain any vertex or any crossing between an edge and $\Lambda$. Edge $(w_1, v_{k+1})$ is drawn as a straight-line segment, while each edge $e_i = (v_{k+1}, w_i)$ $(i = 2, \ldots, h)$ is drawn as a polyline with one bend by suitably choosing two intersection points between $e_i$ and $\Lambda$. The first intersection point is a point of the arc of $\Lambda$ between $v_{k+1}$ and $\beta_{w_1}$, while the second is a point of the left safe region of $w_i$ (i.e., it is a point between $\alpha_{w_i}$ and $w_i$). This choice of the two intersection points guarantees that edges $e_2, e_3, \ldots, e_h$ can be drawn without crossings. For an illustration of the incremental technique described above, see Figure 8.11. For an example of a 1-radial, 1-bend drawing of a planar graph, see Figure 8.12.

**Theorem 8.7** *Every planar graph is* 1*-radial,* 1*-bend drawable.*

A 1-bend drawing on a semi-circle can be seen as an extension of a book embedding on two pages and, indeed, in [DDLW05] planar 1-bend drawings on a semi-circle are used to give an alternative proof of Theorem 8.2. Informally speaking, a planar 1-bend drawing on a semi-

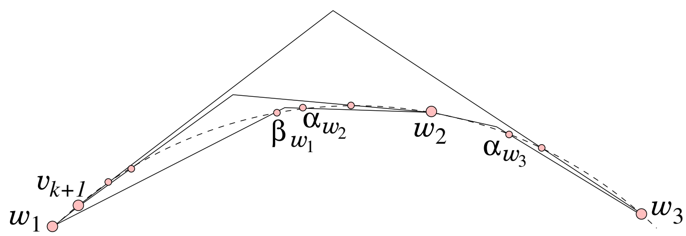



**Figure 8.11** Illustration of the technique used to draw a planar graph on a semi-circle. Figure taken from [DDLW05].

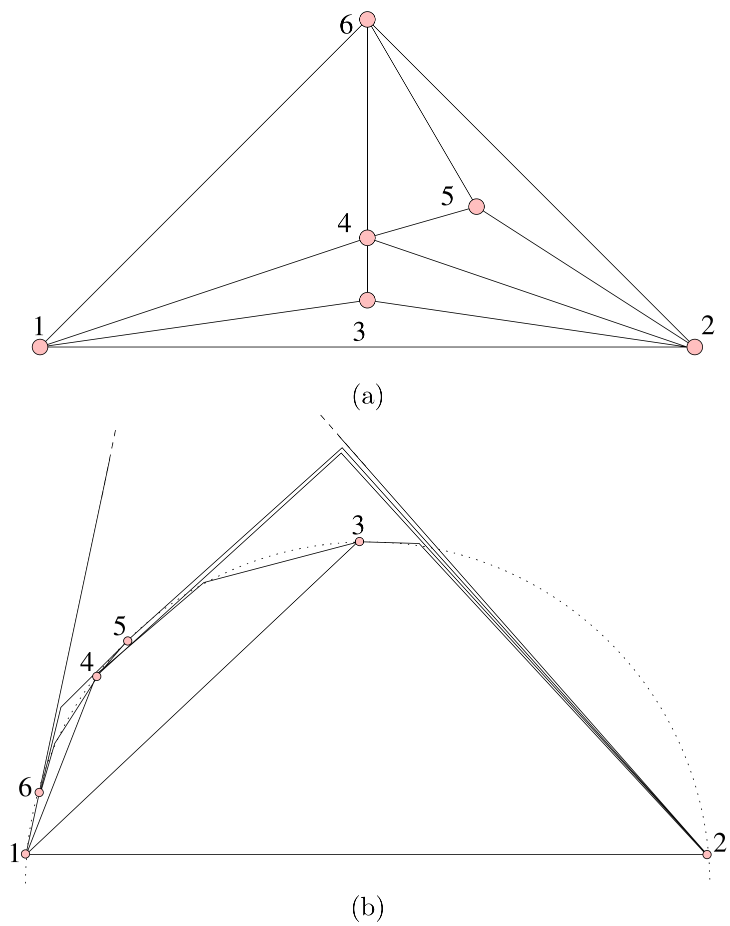



**Figure 8.12** (a) A planar graph $G$ (where the vertices are numbered according to a canonical ordering of $G$). (b) A 1-radial, 1-bend drawing. The linear ordering of the vertices along the (semi)-circle is different from the canonical ordering. Figure taken from [DDLW05].

circle is a topological book embedding where the spine is "bent." By "straightening" this "bent" spine, one can obtain a topological book embedding on two pages. More precisely, according to the algorithm presented in [DDLW05], each edge is either straight-line or it crosses the circle in two points (other than its endvertices). If we consider these two intersection points as two division vertices, then each edge (real or obtained by subdividing a real edge with two division vertices) is either straight-line and completely inside the (semi)-circle or it is bent and completely outside the (semi)-circle. A topological book embedding on two pages can now be computed by assigning edges inside the circle to one page (for example to the one corresponding to the half-plane below the spine), and edges outside the circle to the other page (for example, to the one corresponding to the half-plane above the spine). In the obtained topological book embedding each edge crosses the spine at most twice. However, the 1-bend drawing on a semi-circle is such that one of this spine crossing can be avoided. For an illustration, see Figure 8.13, for more details see [DDLW05].

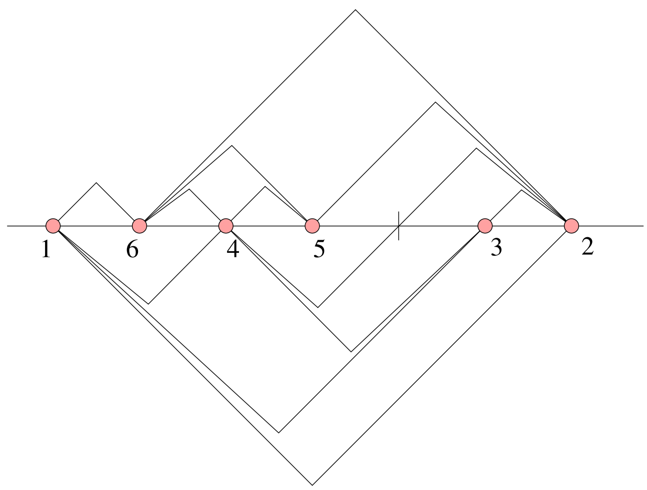



**Figure 8.13** A 1-spine, 2-bend drawing of the graph of Figure 8.12, obtained by using the 1-radial, 1-bend drawing shown in Figure 8.12. Figure taken from [DDLW05].

In [DDLM05] $k$-radial, 0-bend drawings have been studied, for $k \geq 2$. The existence of a $k$-radial, 0-bend drawing of a planar graph $G$ is related to the outerplanarity of $G$. The outerplanarity is defined as follows. A 1-*outerplanar embedded graph* (or simply *outerplanar embedded graph*) is an embedded planar graph where all vertices are on the external face. An embedded graph is a $k$-*outerplanar embedded graph* ($k > 1$) if the embedded graph obtained by removing all vertices of the external face is a $(k-1)$-outerplanar embedded graph. The planar embedding of a $k$-outerplanar embedded graph is called a $k$-*outerplanar embedding*. A graph is $k$-*outerplanar* if it admits a $k$-outerplanar embedding. A planar graph $G$ has *outerplanarity* $k$ (for an integer $k > 0$) if it is $k$-outerplanar but not $(k-1)$-outerplanar. In [DDLM05], it is proved that if a planar graph $G$ admits a $k$-radial, 0-bend drawing, then its outerplanarity is at most $k$. The proof is by induction on the number of circles $k$. If $G$ has a 1-radial, 0-bend drawing, then it is outerplanar by Theorem 8.6. Let $\Gamma$ be a planar $k$-radial, 0-bend drawing of $G$. All the vertices that are on the most external circle in $\Gamma$ are vertices of the external face because the drawing is planar and straight-line. Therefore, removing the vertices of the external face we are left with a $(k-1)$-radial, 0-

bend drawing and, by induction, with an embedded $(k-1)$-outerplanar graph. Therefore, $G$ is an embedded $k$-outerplanar graph and its outerplanarity is at most $k$. In the same paper [DDLM05], an algorithm is presented to compute a $k$-radial, 0-bend drawing of a $k$-outerplanar embedded graph $G$. Figure 8.14 shows an example of a 2-radial, 0-bend drawing of a 2-outerplanar embedded graph. A consequence of these two results in [DDLM05] is that the class of graphs that are $k$-radial, 0-bend drawable, is the class of graphs with outerplanarity at most $k$.

**Theorem 8.8** *A planar graph is $k$-radial, 0-bend drawable ($k > 1$) if and only if its outerplanarity is at most $k$.*

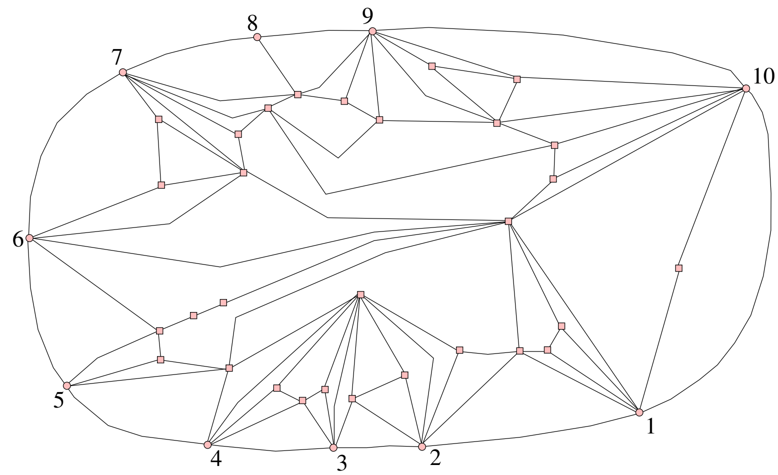
Theorem 8.8 implies that, in order to test whether a planar graph is $k$-radial, 0-bend drawable, one has to compute the outerplanarity of a planar graph $G$. In [DDLM05], it is stated that this can be done in $O(n^5 \log n)$ time based on a result by Bienstock and Monma [BM90]. Recently, this result has been improved to $O(n^4)$ by Angelini et al. [ADP11]; as a consequence the problem of deciding whether a planar graph is $k$-radial, 0-bend drawable can be solved in $O(n^4)$ time. The algorithm by Angelini et al. [ADP11] can also be used to compute a $k$-outerplanar embedding of a planar graph $G$, where $k$ is the outerplanarity of $G$. Thus, another consequence of the results in [DDLM05] is that there exists an $O(n^4)$-time algorithm to compute a $k$-radial, 0-bend drawing of a planar graph $G$ such that $k$ is the minimum possible value. Namely, given a planar graph $G$, one can use the Angelini et al. algorithm to compute a planar $k$-outerplanar embedding of $G$ where $k$ is the outerplanarity of $G$ and then use the algorithm described in [DDLM05] to compute a $k$-radial, 0-bend drawing. The number of circles used is the minimum possible because, if $G$ admitted a $h$-radial, 0-bend drawing for $h < k$, then its outerplanarity would be smaller than $k$.

| | 0 *bends* | 1 *bend* | 2 *bends* |
|---|---|---|---|
| 1 *circle* | outerplanar | all planar [DDLW05] | all planar [DDLW05] |
| $k \geq 2$ *circles* | outerplanarity $\leq k$ [DDLM05] | all planar [DDLW05] | all planar [DDLW05] |

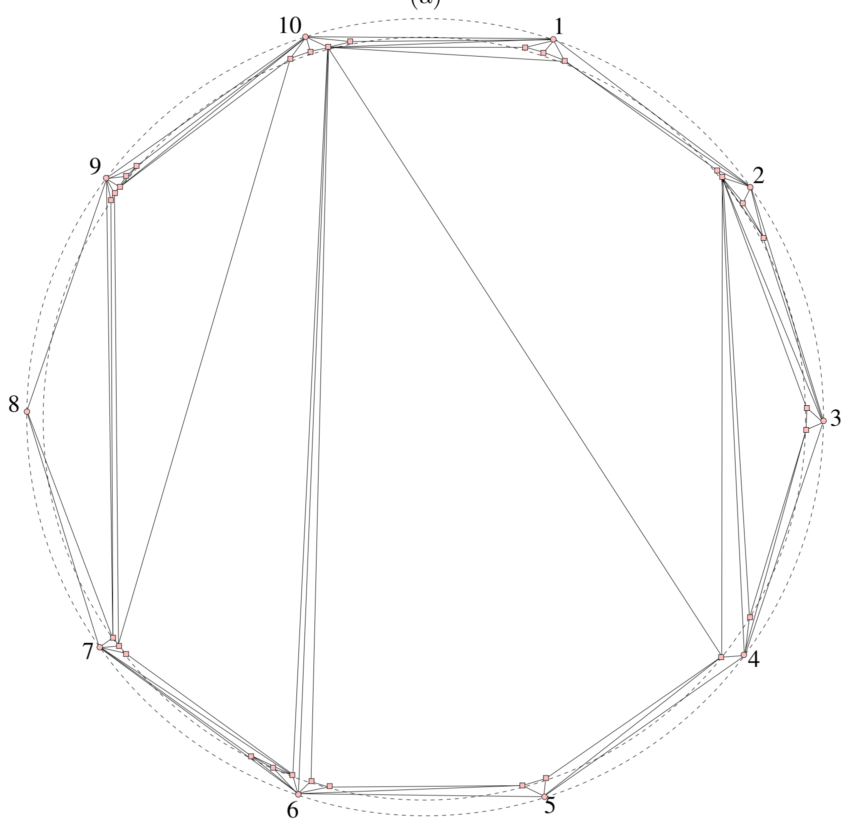
**Table 8.3** Summary of the results about the Characterization Problem for different numbers of circles and bends.

| | 0 *bends* | 1 *bend* | 2 *bends* |
|---|---|---|---|
| 1 *circle* | $O(n)$ | always true | always true |
| $k \geq 2$ *circles* | $O(n^4)$ [ADP11] | always true | always true |

**Table 8.4** Summary of the results about the Drawability Testing Problem for different numbers of circles and bends.




(a)




(b)

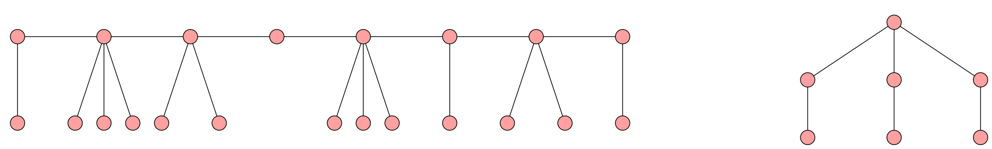
**Figure 8.14** (a) A 2-outerplanar embedded graph $G$. (b) A 2-radial, 0-bend drawing of $G$. Figure taken from [DDLM05].

## 8.4 Results in the Constrained Scenarios

In this section, we describe results about spine and radial drawings with the additional constraints described in Section 8.2.

### 8.4.1 Upright and Proper Spine Drawings

We start by considering upright spine drawings, i.e., drawings where intra-layer edges are not allowed. This constraint implies that the number of layers is at least two, because on a single layer only isolated vertices can be represented. The characterization of upright 2-spine, 0-bend drawable graphs can be stated in several different but equivalent ways. A graph is a *caterpillar* if it consists of a simple path and degree-one vertices attached to this path. A 2-*claw* is a graph consisting of one vertex of degree 3 (the *center*), which is adjacent to three degree-two vertices, each of which is adjacent to the center and to a vertex of degree one. These definitions are illustrated in Figure 8.15. The following characterizations can be found in the works of Eades, McKay and Wormald [EMW86], Harary and Schwenk [HS72], and Tomii, Kambayashi, and Yajima [TKY77].



(a) (b)

**Figure 8.15** (a) A caterpillar. (b) A 2-claw.

**Theorem 8.9** *Let $G$ be a planar graph. The following are equivalent.*

1. *$G$ is upright 2-spine, 0-bend drawable.*
2. *$G$ is a forest of caterpillars.*
3. *$G$ is acyclic and does not contain a 2-claw.*

An interesting work about upright 2-spine, 0-bend drawings is the one by Waterman and Griggs [WG86]. In this paper, the authors study a DNA mapping problem with applications in biology. Very roughly speaking, we have a specific DNA sequence that can be "cut" by means of enzymes. Each cut can be modeled as a partition of a straight line into intervals. Different enzymes give different cuts, i.e., different intervals. Biologists are interested in the order of the "pieces" (intervals) in the sequence, but they cannot directly observe this order. Instead, they can easily establish if different intervals of different cuts (i.e., cuts produced by different enzymes) overlap. This overlapping between intervals can be modeled as a bipartite graph. Namely, let $A$ and $B$ be two cuts of the same DNA sequence. We define a vertex for each interval $a_i \in A$, a vertex for each interval $b_i \in B$, and an edge $(a_i, b_j)$ with $a_i \in A$ and $b_j \in B$ iff $a_i$ and $b_j$ overlap. The problem of reconstructing the two orders of the intervals in $A$ and $B$ can be modeled as the problem of finding an ordering of the vertices in $A$ and an ordering of the vertices in $B$ such that they are "consistent" with the given overlaps. But this means to find a layout of the bipartite graph on two straight lines such that there is no edge crossing. In other words, the problem of reconstructing the two orders of the intervals in $A$ and $B$ is equivalent to the problem of computing an upright 2-spine, 0-bend drawing of the bipartite graph representing the overlaps. Waterman and Griggs study the properties of this bipartite graph, prove that it is a caterpillar and give a linear-time algorithm to compute an upright 2-spine, 0-bend drawing.

Remaining in the case of upright drawings, when the number of spines is greater than two, the problem is different depending on whether one admits long edges (i.e., edges that span more than one level) or not.

In the case when long edges are not allowed, i.e., the case of upright proper drawings[3], Heath and Rosenberg [HR92] show that the drawability testing problem is $\mathcal{NP}$-complete if the number of spines is not fixed. By using the theory of the parametrized complexity, Dujmović et al. [DFK$^+$08] prove that it is possible to decide whether a planar graph $G$ admits an upright proper $k$-spine, 0-bend drawing in $O(f(k) \cdot n)$. This implies that, for a fixed number of layers $k$, $k$-spine, 0-bend drawable graphs can be recognized in linear time. However, the dependency of time complexity from $k$ is given by $f(k) = 2^{32 \cdot k^3}$, which gives impractical large constants also for small values of $k$.

Fößmeier and Kaufmann [FK97] studied upright proper 3-spine, 0-bend drawable graphs, gave a characterization of them, and presented a linear-time algorithm to recognize them. Recently, Suderman [Sud05] pointed out some errors in the work by Fößmeier and Kaufmann and, based on the ideas found there, presented a new characterization and a new linear-time algorithm to recognize upright proper 3-spine, 0-bend drawable graphs. The characterization presented by Suderman consists of constraints on vertices and biconnected components. For example, it is not difficult to see that if $C$ is a biconnected component of an upright proper 3-spine, 0-bend drawable graph, then $G - C$ contains at most two connected components that are not upright proper 2-spine, 0-bend drawable. However, this in itself is not sufficient to guarantee upright proper 3-spine, 0-bend drawability. Consequently, additional constraints must be defined. Suderman describes constraints on vertices and biconnected components that guarantee upright proper 3-spine, 0-bend drawability. Such constraints cannot be easily summarized. The interested reader is referred to the original work by Suderman [Sud05].

Upright spine drawings (proper or not) have also been studied by Suderman [Sud04] in the case of trees with pathwidth $p$. Suderman proves that every tree with pathwidth $p$ admits an upright $k$-spine, 0-bend drawing with $p \leq k \leq \lceil 3p/2 \rceil$ and an upright proper $k$-spine, 0-bend drawing with $p \leq k \leq \lceil 3p-3 \rceil$. Suderman also proves that these bounds are optimal and present linear-time algorithms that, given a tree with pathwidth $p$, compute an upright $k$-spine, 0-bend drawing where $k = \lceil 3p/2 \rceil$ and an upright proper $k$-spine, 0-bend drawing where $k = \lceil 3p - 3 \rceil$. In the same paper [Sud04], Suderman studies proper (non-upright) spine drawings of trees with pathwidth $p$. In this case, a lower bound of $p$ and an upper bound of $2p-1$ on the number of spines in a proper spine drawings of a tree with pathwidth $p$ are given. Also in this case the bounds are optimal and a linear-time algorithm exists to compute a proper $k$-spine, 0-bend drawing with $k = 2p - 1$ of a tree with pathwidth $p$.

[3] These drawings are usually called simply *proper layered drawings*.

### 8.4.2 Partitioned Spine Drawings

As explained in Subsection 8.2.2, in the partitioned layered drawing problem the input graph is partitioned into subsets of vertices, and all vertices in the same set must be drawn on the same layer.

The special case of partitions into two sets have been studied in the literature with two different assumptions: (i) vertices of a same set are never adjacent; (ii) vertices of a same set can be adjacent. Observe that partitioned $k$-spine, 0-bend drawings of a bipartite planar graph with $k \in \{2, 3\}$ can be regarded as upright proper $k$-spine, 0-bend drawings of (non-bipartite) planar graphs. Namely, if a planar graph admits an upright 2-spine, 0-bend drawing, then the vertices on each spine are not adjacent and therefore the graph is bipartite. Analogously, if a planar graph admits an upright proper 3-spine, 0-bend drawing then the vertices on the middle spine are adjacent to the vertices on the top spine and to the vertices on the bottom spine and there is no edge on each spine. This means that the vertices in the middle spine form a set of the bipartition and the vertices in the top and bottom spines form the other set. Thus, the results about upright 2-spine, 0-bend drawings and upright proper 3-spine, 0-bend drawings can also be regarded as results for bipartite graphs.

Biedl [Bie98] characterizes the family of planar graphs that admit a partitioned 2-spine, 0-bend drawing, where vertices in the same set (layer) can be adjacent. Starting from a partitioned planar graph $G = (A \cup B, E)$ Biedl constructs a graph $G'$ whose vertex set is $A \cup B \cup \{v_a, v_b\}$. Vertex $v_a$ is connected to all the edges in $A$, vertex $v_b$ is connected to all the edges in $B$, and $v_a$ and $v_b$ are adjacent. Graph $G$ admits a partitioned 2-spine, 0-bend drawing if and only if $G'$ is planar and there exists a planar embedding of $G'$ such that any triangle containing $v_a$ or $v_b$ is a face (see Figure 8.16).

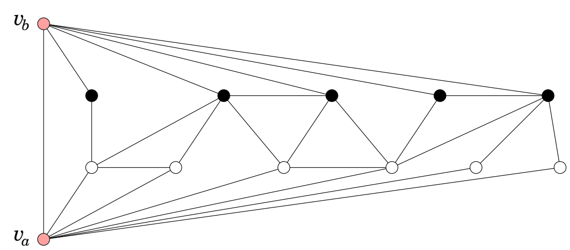



**Figure 8.16** The graph $G'$ constructed by Biedl [Bie98] in order to compute a partitioned 2-spine, 0-bend drawing of a partitioned planar graph $G$.

Partitioned layered drawings on three layers have been studied by Cornelsen et al. [CSW04] who considered partitioned (non-bipartite) planar graphs with the additional property that every $B$-vertex of degree one is adjacent to an $A$-vertex. Cornelsen et al. derive a graph $G'$ from the input graph $G$ by means of a suitable transformation and prove that $G$ admits a partitioned 3-spine, 0-bend drawing if and only if $G'$ admits a 2-spine, 0-bend drawing. Since $G'$ can be computed in linear time and 2-spine, 0-bend drawability can be tested in linear time (see Section 8.3.1), we have that partitioned 3-spine, 0-bend drawable graphs can be recognized in linear time.

We remark that several other models have been introduced in the literature to draw partitioned planar graphs. We recall, for example, the $LH$-drawings, where only one set of the partition is required to be on a straight line while the other is drawn in one of the two half-space defined by the line itself, and the $HH$-drawings, where each set is drawn in one of the two half-planes defined by a straight line. These drawings, however, are not layered drawings, and therefore, we do not describe the results about them here. The interested reader is referred to the literature [Bie98, BKM98].

### 8.4.3 Radial Drawings with Assigned Layers

As discussed in Subsection 8.3.2 for the general scenario, a planar graph is $k$-radial, 0-bend drawable ($k > 1$) if and only if it has outerplanarity at most $k$. The algorithm that computes a $k$-radial, 0-bend drawing strongly relies on the possibility of choosing the radius of each circle, and therefore the distance between every two consecutive layers. This often leads to consecutive layers that are very close to each other, and the angular resolution of the drawing becomes very poor. To improve the readability of radial drawings, consecutive layers should be at least at a given distance that can be specified as part of the input. However, if the layers are given the drawability problem cannot be tackled with the technique described in [DDLM05]. Providing a complete characterization in this case is still an open problem. Partial results are given in [DD03] and in [DGL08]. In [DD03] it is proven that the family of 2-outerplanar embedded graphs whose internal vertices induce a biconnected graph are 2-radial, 0-bend drawable. The drawing can be computed in linear time in such a way that the internal vertices are placed on the internal circle and the external vertices are placed on the external circle. The idea of the drawing technique is as follows (refer to Figure 8.17).

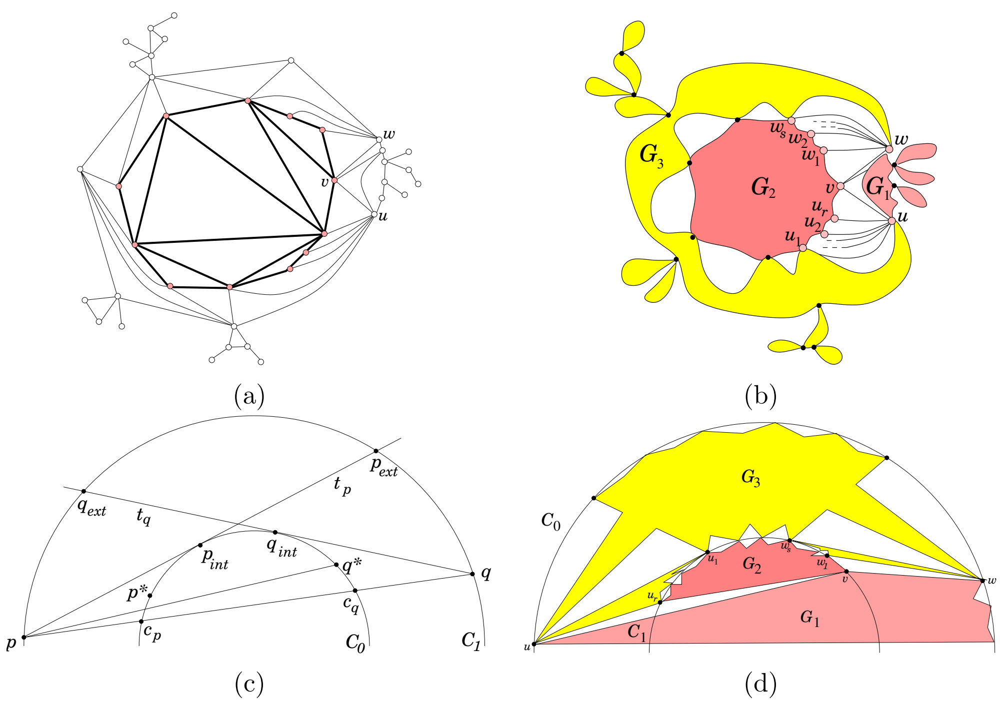



**Figure 8.17** (a) A 2-outerplanar embedded graph $G$ where the internal vertices induce a biconnected graph. (b) The structure of $G$ decomposed into three edge-disjoint outerplanar embedded graphs. (c) Notation used in the description of the drawing algorithm. (d) A 2-radial, 0-bend drawing of $G$ on any two given given circles. Figure taken from [DD03].

Let $G = (V, E)$ be the 2-outerplanar embedded graph given as input, let $C_0$ and $C_1$ be the external and the internal circles given in input, and let $V_0$ and $V_1$ be the external vertices and the internal vertices of $G$. The algorithm places all the vertices on two parallel semi-circles of $C_0$ and $C_1$. First, it chooses two distinct points, $p$ and $q$, of $C_0$ such that: (i) the $x$- and $y$-coordinate of $p$ is less than the $x$- and the $y$-coordinate of $q$, respectively; (ii) segment $\overline{pq}$ is a chord of $C_0$ that has two intersection points $c_p$, $c_q$ with $C_1$, where $c_p$ is the first point encountered while walking on $\overline{pq}$ from $p$ to $q$; (iii) there are two lines $t_p$ and $t_q$ passing for $p$ and $q$, respectively, that are tangent to $C_1$, and intersecting in a point lying in the portion of the annulus delimited by $C_1$ and $C_0$. Denote by $p_{ext} \neq p$ and $p_{int}$ the points where $t_p$ intersects $C_0$ and $C_1$, respectively. Similarly, let $q_{ext} \neq q$ and $q_{int}$ be the points where $t_q$ intersects $C_0$ and $C_1$, respectively. Also denote by $q^*$ any point of $C_1$ between $q_{int}$ and $c_q$, and by $p^*$ any point of $C_1$ between $p_{int}$ and the point $\overline{pq^*} \cap C_1$.

Then the algorithm maps all the vertices of $V_1$ to points of $C_1$, according to the clockwise order they appear on the external boundary of $G(V_1)$, in such a way that: (i) $u_r$ and $u_1$ are mapped to $p^*$ and $p_{int}$, respectively; (ii) $w_s$ is mapped to $q_{int}$; (iii) $v$ is mapped to $q^*$.

Also, it maps all vertices of $V_0$ to points of $C_0$, according to the clockwise order they appear on the external boundary of $G$, in such a way that: (i) $u$ and $w$ are mapped to $p$ and $q$, respectively; (ii) all vertices from $u$ to $w$ are mapped to points between $q_{ext}$ and $p_{ext}$ (iii) all vertices from $w$ to $u$ are mapped to points below $q$.

A characterization of upright 2-radial, 0-bend drawable graphs is given by Di Giacomo et al. [DGL08] who, more in general, studied upright 2-layer, 0-bend drawable graphs in the case when the two layers are two parallel convex curves (a curve is convex if any straight line intersects it in at most two points). The characterization depends on the properties of the curves considered. Roughly speaking, if the two curves have not enough "curvature," then they behave as two straight lines and the class of graphs that admit an upright 2-layer, 0-bend drawing on the two curves coincides with the class of upright 2-spine, 0-bend drawable graphs; on the other hand, if the "curvature" of the two curves is enough, the class of graphs admitting a 2-layer, 0-bend drawing is larger. These concepts can be formalized by defining *paired* and *non-paired* curves (see Figure 8.18). Let $\lambda_e, \lambda_i$ be two parallel convex curves such that the curvature of $\lambda_e$ is less than the curvature of $\lambda_i$; $\lambda_e$ is the *external curve*, $\lambda_i$ is the *internal curve* (in the special case of two concentric circles, $\lambda_e$ is the circle with larger radius). Curves $\lambda_e, \lambda_i$ are *paired* if there exist two points $p \in \lambda_i$ and $q \in \lambda_e$ such that the straight-line segment $\overline{pq}$ intersects $\lambda_i$ twice. Observe that two concentric circles are paired. Two curves will be called *non-paired* if they are parallel, convex, but are not paired. The following theorems are proved in [DGL08].

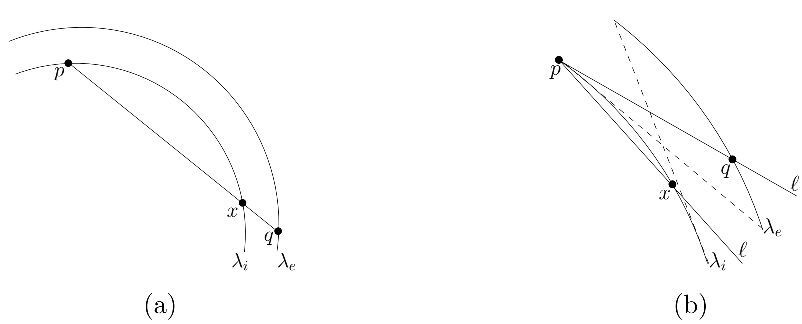
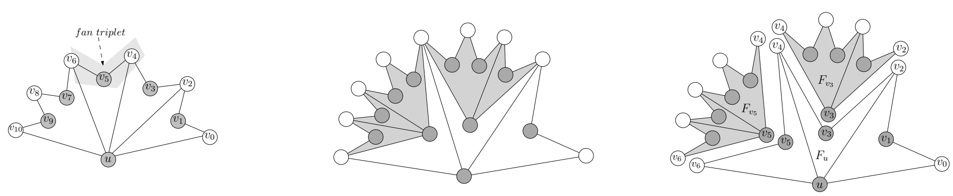



**Figure 8.18** (a) Two paired curves. (b) Two non-paired curves. Figure taken from [DGL08].

**Theorem 8.10** *Let $\mathcal{C}$ be a set of layers consisting of two non-paired curves and let $G$ be a planar graph. $G$ admits an upright 2-layer, 0-bend drawing on $\mathcal{C}$ if and only if $G$ is a forest of caterpillars.*

**Theorem 8.11** *Let $\mathcal{C}$ be a set of layers consisting of two paired curves and let $G$ be a planar graph. $G$ admits an upright 2-layer, 0-bend drawing on $\mathcal{C}$ if and only if $G$ is bipartite and admits a planar embedding such that all vertices of one partite set belong to the external face.*

The proof of Theorem 8.10 is an easy adaptation of the proof of Theorem 8.9. The necessity of Theorem 8.11 can be easily proved as follows. Since the drawing is upright the graph must be bipartite with each partite set defined by the vertices drawn on each curve. Also, since the drawing is straight-line and planar, it defines a planar embedding in which all vertices of the external curve are on the external face. As for the sufficiency, Di Giacomo et al. describe a drawing algorithm based on a suitable decomposition of the graph called *bipartite fan decomposition*. A bipartite fan is a biconnected bipartite planar graph having a vertex $u$, called *apex*, that is shared by all its faces (including the external one). Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of a fan $G$ in the counterclockwise order they have on the external face. Any three vertices $v_{2j}, v_{2j+1}, v_{2j+2}$ $(0 \leq j \leq \frac{n-4}{2})$ form a *fan triplet* of $G$. Notice that $v_{2j+1}$ belongs to the same partite set as $u$. See Figure 8.19 (a) for an example of a bipartite fan.



(a) (b) (c)

**Figure 8.19** (a) A bipartite fan. (b) A bipartite graph $G$ embedded with all vertices of one partition set on the external face. (c) A bipartite fan decomposition of $G$. Figure taken from [DGL08].

Given a biconnected bipartite graph embedded with all vertices of one partition set on the external face,[4] it is possible to decompose it into bipartite fans as follows. A first bipartite fan $F_u$ is computed; the two edges of each fan triplet of $F_u$ either belong to the external face, or they are a cut-set for $G$ and they identify a subgraph that can be recursively decomposed (see Figure 8.19 (b) for an example of bipartite fan decomposition). Once $G$ has been decomposed, a wedge $W_u$ is defined on the paired curves; a wedge is a portion of plane delimited by the external curve $\lambda_e$ and by two segments having and endpoint on each curve, one of which has two intersections with the internal curve $\lambda_i$ (see Figure 8.20 (a) for an example). Fan $F_u$ is drawn inside $W_u$ as shown in Figure 8.20 (b). Notice that the drawing of $F_u$ is such that each fan triplet defines a new wedge where the subgraph

[4]If the input graph is not biconnected, it can be augmented with vertex and edge addition to became biconnected while maintaining all the vertices of one partition set on the external face. For details, see [DGL08]

identified by the fan triplet can be recursively drawn. We conclude by mentioning that based on Theorem 8.11 upright 2-layer, 0-bend drawable graphs can be recognized in linear time and that, when the two paired curves are two circles, an upright 2-radial, 0-bend drawing can be computed in linear time [DGL08].

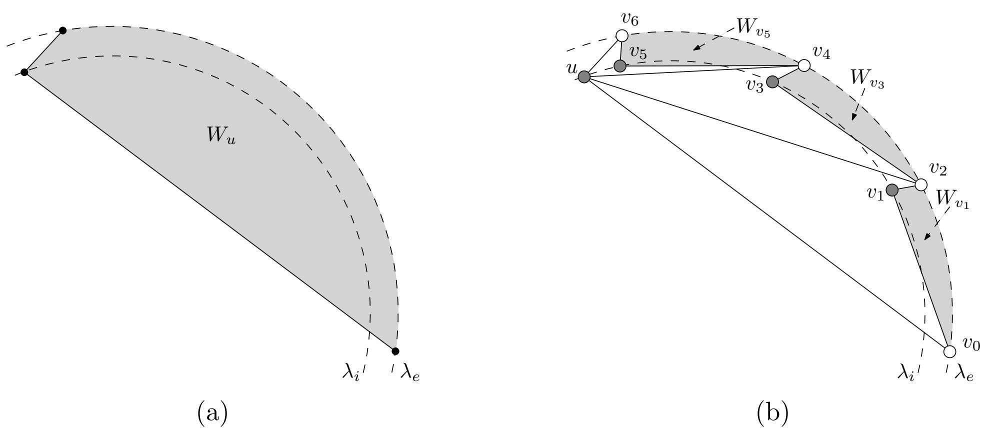



**Figure 8.20** (a) A wedge $W_u$ defined on two paired curves. (b) A 2-layer, 0-bend drawing of the fan $F_u$ of Figure 8.19 inside $W_u$.

Di Giacomo et al. [DDL08b] studied $k$-radial drawings of graphs with assigned layers and a prescribed assignment of vertices to the layers. More precisely, the layers are concentric circles such that the difference between the radii of any two consecutive circles is constant and equal to the radius of the smallest circle. Also, a function $\phi : V \to \{0, 1, \ldots, k-1\}$ is given and it is required that each vertex $v \in V$ is drawn as a point of circle $C_{\phi(v)}$. A planar graph $G$ equipped with such a function is called a *layered planar graph*. We observe that the assignment of vertices to layers described by the function $\phi$ represents a stronger constraint than assigning a vertex partition. In [DDL08b], $k$-radial drawings with different trade-offs between the maximum number of bends along an edge and the *angular distance ratio* are studied. The angular distance ratio measures how uniform is the angular distribution of the vertices. More precisely, let $\rho_0, \rho_1, \ldots, \rho_{h-1}$ ($h \geq 1$) be the distinct rays passing through the vertices in the order they are encountered in a radial sweep of the drawing. If $h > 1$, define $\alpha_i = (\angle\rho_{i+1} - \angle\rho_i)$ (the indices are taken modulo $h$ and the angles are measured modulo $2\pi$), $\alpha_{min} = \min_i\{\alpha_i\}$ and $\alpha_{max} = \max_i\{\alpha_i\}$. If $h = 1$, we define $\alpha_{min} = 0$ and $\alpha_{max} = 2\pi$. The angular distance ratio is defined as $ADR = \frac{\alpha_{max}}{\alpha_{min}}$. Notice that, when $h = 1$ we have $ADR = +\infty$.

In [DDL08b], it is first proved that there exist layered graphs that do not admit a $k$-radial, 0-bend drawing satisfying the vertex assignment that have optimal angular distance ratio (i.e., $ADR = 1$). The graph $G = (V, E, \phi)$ is defined as follows (refer to Figure 8.21). The set of vertices is $V = \{u_0, u_1, \ldots, u_{h-1}\} \cup \{v_0, v_1, \ldots, v_{h-1}\} \cup \{w_0, w_1, \ldots, w_{h-1}\}$ with $h \geq 3$; the set of edges is $E = \{(u_i, u_{i+1}), (v_i, u_i), (v_i, u_{i+1}), (w_i, u_i), (w_i, u_{i+1}), (w_i, v_i) \mid 0 \leq i \leq h-1\}$ (indices are taken modulo $h$), $\phi(u_i) = 0$, $\phi(v_i) = 0$, and $\phi(w_i) = 1$ ($i = 0, \ldots, h-1$). Consider now a 3-cycle $u_i, u_{i+1}, v_i$ ($i = 0, \ldots, h-2$). All the vertices of the cycle must be drawn on circle $C_0$ and if we want $ADR = 1$ the angle between the two rays passing through $u_i$ and $u_{i+1}$ must be $\frac{2\pi}{h}$. Vertex $w_i$ must be drawn on circle $C_1$ and, in order to guarantee planarity, $w_1$ must be inside the triangle representing the 3-cycle $u_i, u_{i+1}, v_i$. It follows that circle $C_1$ must cross the segment representing the edge $(u_i, u_{i+1})$; thus, it must be $r_1 \geq r_0 \cos(\frac{\pi}{h})$, but this is possible only if $h < 3$ because $r_1 = \frac{1}{2} r_0$.

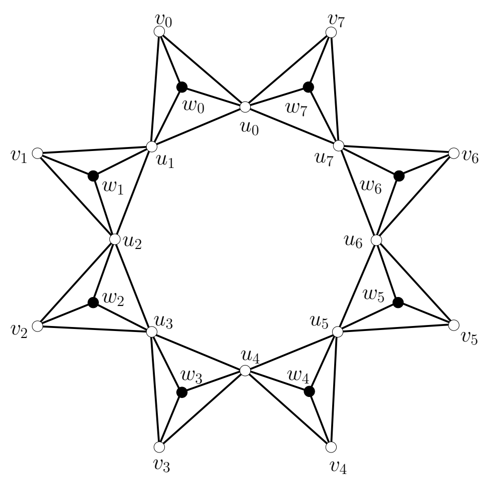



**Figure 8.21** A layered planar graph that does not admit a $k$-radial, 0-bend drawing with optimal angular distance ratio if the white vertices are assigned to layer 0 and the black vertices are assigned to layer 1. Figure taken from [DDL08b].

The negative result above motivates the study of $k$-radial drawings with bends. Di Giacomo et al. [DDL08b] prove that every layered planar graph $G$ admits a $k$-radial, 3-bend drawing consistent with the assignment of the vertices the layers having optimal angular distance ratio. Such a drawing can be computed in linear time. It is interesting to note that the drawing algorithm exploits the connection between 1-spine, 2-bend drawings, topological book embeddings, and Hamiltonicity observed in Section 8.3.1 and that will be explained in detail in Section 8.5.1. By using the Hamiltonian augmentation technique described in [DDLW05], a Hamiltonian augmentation $\mathsf{Ham}(G)$ of $G$ and an augmenting Hamiltonian cycle $H$ of $G$ are computed. The cycle $H$ is drawn with straight-line edges (and each vertex $v$ drawn on circle $C_{\phi(v)}$). All the remaining edges are either inside $H$ or outside it in the planar embedding of $\mathsf{Ham}(G)$. The edges that are outside $H$ are drawn as a 2-bend polyline outside the polygon representing $H$; the edges that are inside $H$ are drawn as a 1-bend polyline inside the polygon representing $H$. The properties of the cycle $H$ computed with the augmentation technique of [DDLW05] guarantees that edges subdivided with a division vertex have at most three bends.

In [DDL08b], a drawing algorithm to compute a $k$-radial, 2-bend drawing consistent with the assignment of the vertices to the layers is also presented. In this case, however, the angular distance ratio is not optimal.

## 8.5 Related Problems

In this section, we present two applications of the results described in Subsections 8.3.1 and 8.3.2. The first application is in the field of graph theory and the second one is in computational geometry.

### 8.5.1 Hamiltonicity

We have already seen in the description of Section 8.3.1 that there is a connection between 1-spine, 1-bend drawings and Hamiltonicity. As stated by Theorem 8.1, a planar graph admits a 1-spine, 1-bend drawing if and only if it is sub-Hamiltonian. Given a planar 1-spine, 1-bend drawing of a planar graph $G$ (or equivalently a book embedding on two pages), denote by $v_0, v_1, \ldots v_{n-1}$ the vertices of $G$ in the order they appear along the spine. $\mathsf{Ham}(G)$ can be computed by augmenting $G$ with the edges $(v_i, v_{i+1})$ (indices are taken modulo $n$) that are not in $G$. A Hamiltonian cycle of $\mathsf{Ham}(G)$ is given by the sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_0)$. This implies that if one can compute a planar 1-spine, 1-bend drawing (or equivalently a book embedding on two pages) efficiently, then it is also possible to find an augmenting Hamiltonian cycle of $G$ efficiently. Since a book embedding on at most two pages can be computed in $O(n)$ time for outerplanar graphs [BK79], series-parallel graphs [DDLW06], planar bipartite graphs [ddMP95], square grids and $X$-trees [CLR87], for all these families of graphs it is also possible to find an augmenting Hamiltonian cycle in $O(n)$ time.

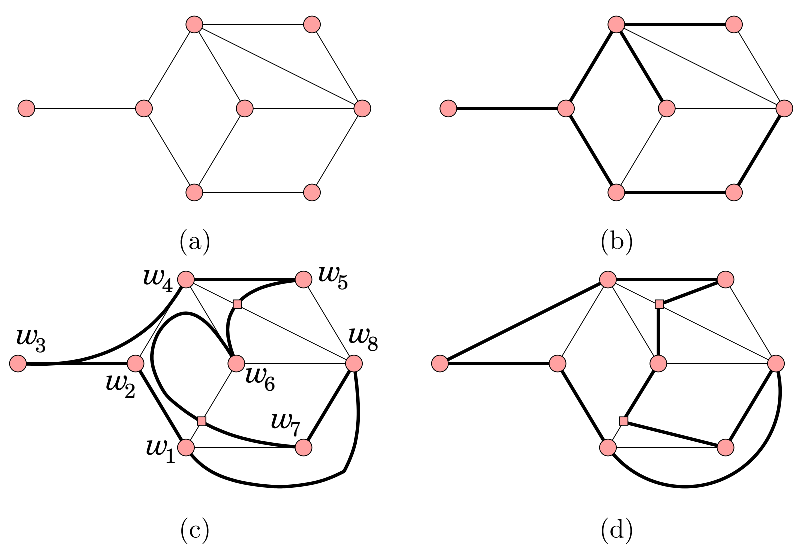



**Figure 8.22** The PW Augmentation Technique [PW01]. (a) A planar graph $G$. (b) A spanning tree $S$ of $G$. (c) Visit of $S$. (d) The resulting Hamiltonian augmentation of $G$. Figure taken from [PW01].

In the general case of planar graphs, there are different techniques to compute $\mathsf{Ham}(G)$ and a Hamiltonian cycle of $\mathsf{Ham}(G)$, i.e., an augmenting Hamiltonian cycle of $G$. A first technique is the one described by Pach and Wenger [PW01] (see also Figure 8.22), which we will call the *PW Augmentation Technique*. Let $S$ be a spanning tree of $G$ and let $\Gamma$ be a planar drawing of $G$. Starting at any vertex, walk clockwise around $S$, visiting its vertices in order. Note that the internal vertices of $S$ will be visited more than once. Label the vertices with $w_1, w_2, \ldots, w_n$ by the order in which they are first visited. If $w_i$ and $w_{i+1}$ are connected by an edge, then let this edge belong to the Hamiltonian cycle ($1 \leq i \leq n$ and assume that the indices are taken modulo $n$). If not, connect $w_i$ to $w_{i+1}$ by a simple curve clockwise around the boundary of $S$, passing very close to it. Wherever this curve intersects an edge of $G$, introduce a new vertex. This curve becomes a path whose pieces are added as edges to the graph and to its Hamiltonian cycle. Multiple edges (if any) are merged and the resulting graph is $\mathsf{Ham}(G)$. It can be proved that, using the PW Augmentation Technique, each edge is split at most twice. Since $G$ has at most $3n-6$ edges and the edges of $S$ are not split, $\mathsf{Ham}(G)$ has at most $5n-10$ vertices.

An alternative technique to compute a Hamiltonian augmentation of $G$ is the one described in the work by Kaufman and Wiese [KW02], which we will denote as the *KW Augmentation Technique*. This technique is based on the fact that every 4-connected graph is Hamiltonian and a Hamiltonian cycle can be found in $O(n)$ time [CN89]. Thus, the idea of Kaufman and Wiese is to make a graph 4-connected. Assume that the input graph is maximal planar (if not it can be augmented in linear time to a maximal planar graph). By using an algorithm by Chiba and Nishizeki [CN85] one can find the separating triangles of $G$ in $O(n)$ time. Each separating triangle can be removed by using the following approach. Let $e=(u,v)$ be and edge of a separating triangle. Since $G$ is maximal planar, $e$ is shared by two triangular faces $u,v,w$ and $u,v,z$. Edge $e$ is replaced by a chain consisting of two edges $(u,d)$, $(d,v)$ and a division vertex $d$. Furthermore, edges $(d,w)$ and $(d,z)$ are added to the graph (see Figure 8.23). By applying this transformation the separating triangle has been removed and no other separating triangle is created. Thus repeatedly applying this technique for every separating triangle we eventually obtain a 4-connected graph, which therefore is a Hamiltonian augmentation $\mathsf{Ham}(G)$ of $G$. The algorithm by Chiba and Nishizeki [CN89] can then be applied to find a Hamiltonian cycle in $\mathsf{Ham}(G)$. The KW Augmentation Technique splits each edge with at most one division vertex, therefore $\mathsf{Ham}(G)$ has at most $4n-6$ vertices.

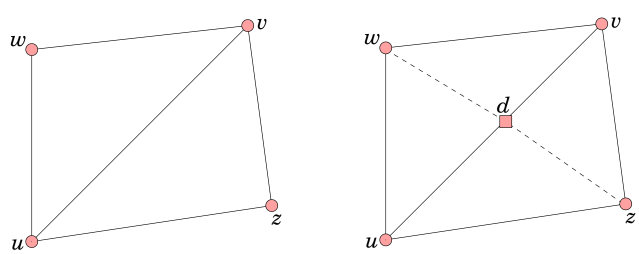
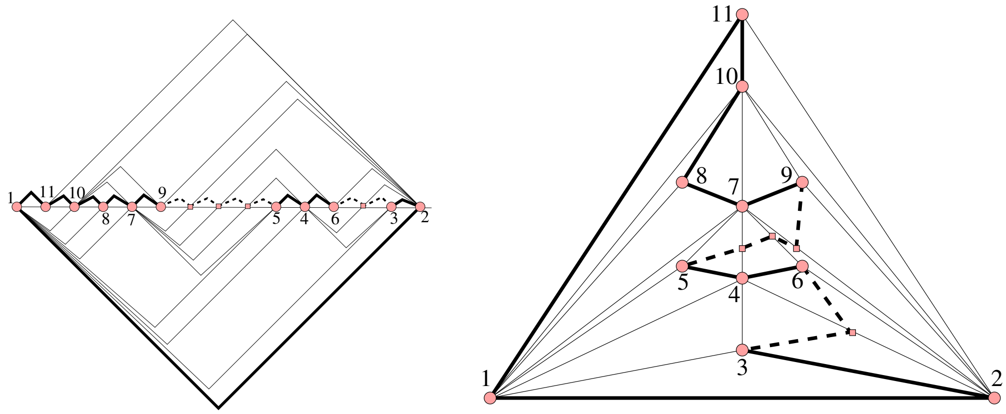



**Figure 8.23** The augmentation described by Kaufmann and Wiese [KW02] to make a planar graph 4-connected.

The curve embedding defined in [DDLW05] can be used to define another alternative technique to compute a Hamiltonian augmentation of $G$, which will be called in the following the *DDLW Augmentation Technique*. As explained in Section 8.3.2, a curve embedding of a planar graph is a linear ordering $L$ of the vertices of $G$ such that $G$ admits a planar 1-bend drawing on any concave curve $\Lambda$ where the vertices appear along $\Lambda$ in the same order as $L$. In particular, such an ordering can be computed by drawing $G$ on a semi-circle with at most 1 bend per edge according to the technique described in Section 8.3.2. As already explained in Section 8.3.2, by using the 1-bend drawing on a semi-circle we can obtain a topological book embedding of $G$ on two pages where each edge crosses the spine at most once. If we consider the crossings between the edges and the spine as division vertices of the edges, we have a book embedding on two pages of a subdivision $\mathsf{sub}(G)$ of $G$. Graph $\mathsf{sub}(G)$ has at most one division vertex per edge. Since $\mathsf{sub}(G)$ admits a book embedding on two pages, it is sub-Hamiltonian and we can augment it with edge addition so to make it Hamiltonian. As explained above this can be done by adding edges between non adjacent vertices that are consecutive along the spine of the book embedding and between the first and the last vertex on the spine if such an edge does not exist (see Figure 8.24). With the DDLW Augmentation Technique, each edge is split at most once and therefore, like in the case of the KW Augmentation Technique, $\mathsf{Ham}(G)$ has at most $4n-6$ vertices. However, the DDLW Augmentation Technique does not require to preliminarily make $G$ 4-connected. The augmenting Hamiltonian cycle $H$ of $G$ computed by the DDLW Augmentation Technique has another interesting property. Let $d$ be a division vertex that subdivide the edge $(u, v)$, and consider the linear ordering of both the real vertices and the division vertices defined by the topological book embedding of $\mathsf{sub}(G)$ used to compute $\mathsf{Ham}(G)$. The division vertex $d$ is encountered after $u$ and before $v$ in the considered order. This is a consequence of the fact that, according to the algorithm described in [DDLW05], the crossing between an edge and the spine always falls between the end-vertices of the edge. We say that all the division vertices of $H$ are *flat* with respect to the considered order. The flatness of the division vertices will be used in the next application.




(a) (b)

**Figure 8.24** (a) A 1-spine, 2-bend drawing (or equivalently a topological book embedding on two pages) of the non-Hamiltonian graph $G$ of Figure 8.4 obtained by using the curve embedding. (b) A Hamiltonian augmentation of $G$.

### 8.5.2 Point-Set Embeddability

The results described in Subsections 8.3.1 and 8.3.2 can be applied to the point-set embedding problem, which is widely investigated both in graph drawing and in computational geometry. Let $G$ be a planar graph with $n$ vertices and let $S$ be a set of $n$ points in the plane, a *point-set embedding* of $G$ onto $S$ is a planar drawing of $G$ such that each vertex of $G$ is represented by a point of $S$. Observe that there are two main variants of this problem, depending on whether the mapping between the vertices and the points is given as a part of the input or not. If the mapping is not given and the points are in general position, then every outerplanar graph admits a point-set embedding on any given set of points and straight-line edges [Bos02]. In [Bos02], an $O(n \log^3 n)$-time algorithm is also presented to compute a straight-line point-set embedding of an outerplanar graph $G$ on a given set of points. For trees, an optimal $\Theta(n \log n)$-time algorithm is given by Bose et al. [BMS97], who improve previous results by Ikebe et al. [IPTT94] and Pach and Törőcsik [PT93].

The problem of deciding whether there exists a point set embedding with straight-line edges of a planar graph on a given set of points is, in general, $\mathcal{NP}$-hard [Cab06]. Since outerplanar graphs are the largest class of graphs admitting a straight-line point-set embedding on *every* set of points [GMPP91] in general position, Kaufmann and Wiese [KW02] investigate the problem of computing a point-set embedding of a planar graph with a small number of bends per edge. They show that any planar graph admits a point-set embedding with at most two bends per edge on any given set of points, and that two bends are required in some cases. Pach and Wenger [PW01] show that, if the mapping of the vertices of $G$ to the points of $P$ is given, then a planar drawing of $G$ exists with $O(n)$ bends per edge and that $\Omega(n)$ bends per edge may be necessary even for paths. Recently, the two main variants (with or without mapping) have been unified and generalized by introducing the concept of coloured point-set embedding where the set of vertices and the set of points are coloured with $k$ colours and it is required that each vertex is drawn on a point with the same colour [BDL08, DDL$^+$08a, DLT06, DGLT10]. Badent et al. [BDL08] generalized the result by Pach and Wenger by proving that, for every $k \geq 2$, a $k$-coloured planar graph admits a $k$-coloured point-set embedding on every $k$-coloured set of points with $O(n)$ bends per edge. They also show that $\Omega(n)$ bends may be necessary.

We briefly recall here the technique of Kaufmann and Wiese [KW02] and highlight connections between this technique and spine drawings. Assume first that the input graph $G$ is (sub)-Hamiltonian. Let $H = v_1, v_2, \ldots, v_n$ be a (augmenting) Hamiltonian cycle in $G$, and let $\Psi$ be a planar embedding of $G$ such that edge $(v_1, v_n)$ lies on the external face. Let $p_1, p_2, \ldots, p_n$ be the sequence of points in $S$ ordered by increasing $x$-coordinates (we can assume that all the points have distinct $x$-coordinates because, if not, we can rotate the plane to achieve this condition). Assign each vertex $v_i$ to point $p_i$ in $P$ and draw the edges of path $P = H \setminus \{(v_1, v_n)\}$ as straight-line segments. Draw each remaining edge $e$ using two segments, one with slope $\sigma > 0$ and the other with slope $-\sigma$. In order to prevent $e$ from crossing the previously drawn edges, the slope $\sigma$ is chosen to be greater than the absolute value of the slope of each edge in $P$. With segments of slope $\pm\sigma$, it is possible to draw $e$ above or below $P$. Edge $e$ is drawn above $P$ if $e$ is on the left-hand side when walking from $v_1$ to $v_n$ in $G$, and below $P$ otherwise. The resulting drawing is planar except that edges outside $P$ incident to the same vertex may contain overlapping segments. To eliminate overlapping, perturb overlapping edges by decreasing the absolute value of their segment slopes by slightly different amounts (see [KW02] for details).

When the input graph $G$ is not Hamiltonian, Kaufmann and Wiese compute a Hamiltonian augmentation $\mathsf{Ham}(G)$ of $G$ by using the KW Augmentation Technique described in Section 8.5.1. Since $\mathsf{Ham}(G)$ has more vertices than $G$, the set of points $S$ is also enriched with extra points at suitable positions. $\mathsf{Ham}(G)$ can be point-set embedded as described above. Some edges of $G$ are split into two pieces in $\mathsf{Ham}(G)$. Let $e = (u, v)$ be an edge of $G$ split by a division vertex $d$ in $G'$. The edge $e$ is replaced by the two edges $(u, d)$ and $(d, v)$; each of these two edges may have one bend. Furthermore, the two segments incident to $d$ can have different slopes, thus creating a third bend at $d$. Hence, each edge of $G$ is drawn with at most three bends. In order to remove the third bend, Kaufmann and Wiese rotate the segments incident to $d$ and make them both vertical. Note that this may imply to rotate other segments that are "above" or "below" the rotating segments. An example of a point-set embedding of the non-Hamiltonian graph $G$ of Figure 8.4 computed by the Kaufmann and Wiese technique is shown in Figure 8.25.

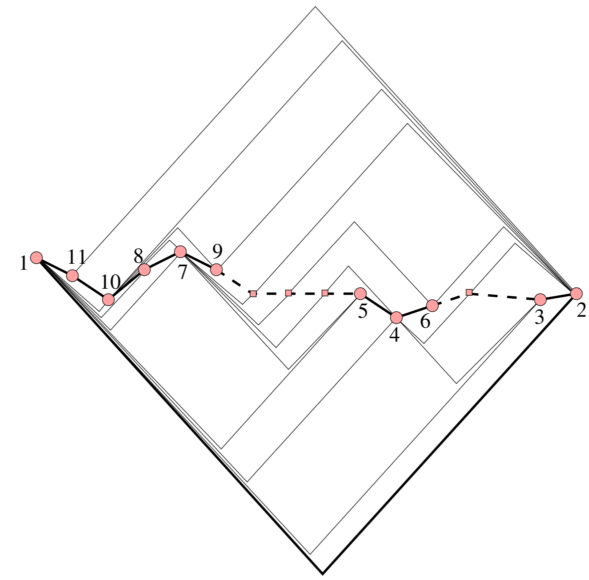



**Figure 8.25** A point-set embedding of the non-Hamiltonian graph $G$ of Figure 8.4. The drawing is created with the Kaufmann and Wiese technique [KW02] and using the Hamiltonian cycle (highlighted in the picture) shown in Figure 8.24.

As one can see from the description above, computing an augmenting Hamiltonian cycle of the input graph $G$ plays an important role in the Kaufmann and Wiese technique. As discussed in Subsection 8.5.1, Hamiltonicity is related to spine and radial drawings. A first consequence of this fact is that one can compute a point-set embedding on any set of points with at most 1 bend per edge for all those families of (sub)-Hamiltonian graphs for which a (augmenting) Hamiltonian cycle can be found efficiently. In Section 8.5.1, we have seen that among these families we have outerplanar graphs, series-parallel graphs, planar bipartite graphs, square grids, and $X$-trees.

Another connection between spine and radial drawings, Hamiltonicity, and point-set embeddings, is given by the fact that one can use the DDLW Augmentation Technique (see Section 8.5.1) as an alternative to the KW Augmentation Technique to compute a Hamiltonian augmentation of the input graph $G$. The DDLW Augmentation Technique has the advantage that the rotation needed to avoid the third bend is not required. If $e = (u, v)$ is an edge of $G$ split by a division vertex $d$, the rotation is needed only when the $x$-coordinate of $d$ is not between the $x$-coordinates of $u$ and $v$, i.e., only if $d$ is not flat with respect to the left to right order of the vertices (see Figure 8.26). As pointed out in Section 8.5.1, the technique based on curve embeddings guarantees that $d$ is always flat, and thus no rotation is required. To avoid the final rotation not only simplifies the drawing algorithm, but it also has impact on the area of the final drawing. Namely, Kaufmann and Wiese prove that the drawing before the rotation has area $O(W^3)$, where $W$ is the *size* of $S$, i.e., the length of the side of the smallest axis parallel square containing $S$. The rotation may cause an exponential growth of the area of the drawing. Thus, avoiding the rotation keeps the drawing in a polynomial area.

We conclude this section by mentioning that the DDLW Augmentation Technique has been used to investigate other problems related to point-set embeddability, such as the study of universal point sets. A set $S$ of $m$ points is *h-bend universal* for a family of planar

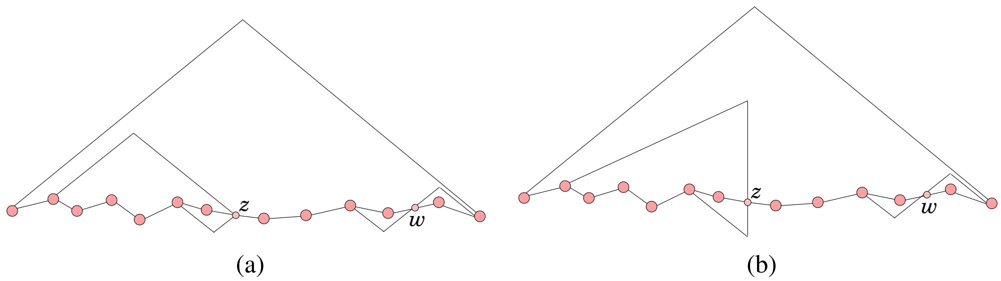



**Figure 8.26** An illustration of the segments rotation performed in the technique by Kaufmann and Wiese [KW02] in order to remove a third bend. The division vertex $z$ requires the rotation, the division vertex $w$ does not require the rotation. Figure taken from [DDLW05].

graphs with $n$ vertices ($n \leq m$) if each graph in the family admits a point-set embedding on a subset of $S$ that has at most $h$ bends per edge.

Many results about point-set embeddings can be regarded as results about universal point sets. For example, the results about the point-set embeddability of outerplanar graphs on every set of points in general position [Bos02] imply that every set of points in general position is 0-bend universal for the class of outerplanar graphs with $n$ vertices. Analogously, the result about Kaufmann and Wiese implies that every set of points is 2-bend universal for the class of planar graphs.

De Fraysseix, Pach, and Pollack [dPP90] and independently Schnyder [Sch90] proved that a grid with $O(n^2)$ points is 0-bend universal for all planar graphs with $n$ vertices. De Fraysseix et al. [dPP90] also showed that a 0-bend universal set of points for all planar graphs having $n$ vertices cannot have $n+o(\sqrt{n})$ points. This last lower bound was improved by Chrobak and Karloff [CK89] and later by Kurowski [Kur04] who showed that linearly many extra points are necessary for a 0-bend universal set of points for all planar graphs having $n$ vertices.

Since 0-bend universal point sets for planar graphs must have more that $n$ points [Kur04], while every set of $n$ points is 2-bend universal for planar graphs [KW02], Everett et al. [ELLW10] investigated 1-bend universal point sets and proved that there exists a set of $n$ distinct points in the plane in general position that is 1-bend universal for all planar graphs with $n$ vertices. The proof of the latter result is constructive. A set $S$ of $n$ points is defined and a point-set embedding of a planar graph $G$ on this set of points is constructed by exploiting the DDLW Augmentation Technique. Namely, the points are chosen to be in convex position and an augmenting Hamiltonian cycle $H$ of the input graph $G$ is drawn as the convex hull $CH$ of $S$ suitably enriched with extra points that represent the division vertices. The edges of $\mathsf{Ham}(G)$ that are not in $H$ are either inside $H$ or outside it. Those inside are drawn as chords inside $CH$, the others are drawn with one bend outside $CH$. The choice of points and the property of $H$ that all division vertices are flat guarantee that no additional bend is required when the division vertices are removed.

Dujmović et al. [DEL$^+$13] study 0-bends universal point sets for sub-classes of planar graphs. They prove that there exist sets of $n$ points that are 0-bend universal for maximum degree 3 series-parallel lattices with $n$ vertices. They also study $h$-bend universal point sets with the additional requirement that bends are also constrained to be represented by points in the set. They prove that, if 1, 2, or 3 bends per edge are allowed then universal point sets exist of size $O(n^2/\log n)$, $O(n \log n)$, and $O(n)$, respectively. All these results use as a basic tool the DDLW Augmentation Technique.

## 8.6 Conclusions

In this chapter, layered drawing conventions and drawing algorithms have been presented, where layers can be parallel straight lines (spine drawings) or concentric circles (radial drawings). One of the main differences between these drawings and hierarchical drawings is that we do not take into account the orientation of the edges and we do not require that edges are represented as monotone curves in a common direction.

In the discussion of the results, we used a unified framework for spine and radial drawings, which studies the drawability problem assuming that upper bounds are given on the number of layers and on the number of bends along each edge. We summarized the literature by providing characterization and time-complexity results for each specific drawability problem, and we also presented variations of the problem and related results for some constrained scenarios.

Some theoretical connections between spine drawings, radial drawings, and well-studied problems in graph theory and computational geometry were also pointed-out.

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