4

Proximity Drawings

Giuseppe Liotta
University of Perugia

4.1 Introduction

In 1969, Gabriel and Sokal [GS69] presented a method for associating a graph to a set of geographic data points P by connecting points $x, y \in P$ with an edge if and only if the closed disk having the segment $\overline{xy}$ as diameter contained no other point of P. This geometric graph, called the *Gabriel graph* of P, is just one example of what have come to be called *proximity graphs.* Loosely speaking, a proximity graph is a geometric graph (i.e., a straight-line drawing) constructed from a set P of points in some metric space by connecting pairs of points that are deemed to be "sufficiently" close together. A set P can give rise to a variety of different proximity graphs depending upon the definition of closeness used.

Proximity graphs have applications in numerous areas where they are commonly used to describe the underlying "shape" of a set of points, including computer graphics, computational geometry, pattern recognition, computational morphology, numerical analysis, computational biology, and GIS (see, e.g., [OBS92, GO04]). A paper by Toussaint [Tou05] describes applications of proximity graphs in the context of instance-based learning and data-mining and a paper by Carreira-Perpinan and Zemel [CPZ04] to the field of clustering and manifold learning. Motivated by these many applications, a rich body of computational geometry literature has been devoted to the question of efficiently computing different types of proximity graphs of a given set of points. For exhaustive lists of references on the subject, the interested reader is referred to the above-mentioned paper by Toussaint [Tou05] and to the survey by Jaromczyk and Toussaint [JT92].

In this chapter, we shall look at proximity from the different perspective of graph drawing: The goal is computing a straight-line drawing of a given graph with the additional constraint that the drawing be a proximity graph. There is a strong connection between the graph drawing and the computational geometry point of views about proximity. Indeed, the (computational geometry) problem of analyzing the combinatorial properties of a given type of geometric graph naturally raises the (graph drawing) question of characterizing those graphs that admit the given type of straight-line drawing. This in turn leads to the investigation of the design of efficient algorithms for computing such a drawing when one exists.

We therefore will talk about the *proximity drawability problem*: Given a graph G and a definition of proximity, determine whether a set P of points exists such that the proximity graph of P is isomorphic with the given graph, and if so, compute such a set. Clearly, the set P, if it exists, gives rise to a straight-line drawing of G, called a *proximity drawing* of G, where each vertex of G is mapped to a distinct point of P and each edge to a straight-line segment between pairs of points of P. Proximity drawings have several interesting features. They are usually unaffected by changes in scale, since the measures of proximity used are based on relative distances between points. Also, adjacent vertices are drawn (relatively) more closely together than non-adjacent vertices, and vertices not incident to a particular edge are not drawn too close to the edge. Furthermore, neighbors of a given vertex tend to cluster together.

This chapter surveys some of the central problems, results, and research trends on proximity drawings. Although many of the ideas described here can be developed in the more general setting of a metric space, we shall most often assume that the drawings are to be made in Euclidean d-space (the only exception will be for Voronoi and Delaunay drawings).

The remainder of this chapter is structured as follows. In Section 4.2, various definitions of proximity drawings are given; in Section 4.3, the basic graph drawing literature on the proximity drawability problem is reviewed. Section 4.4 introduces extensions and relaxations of the definition of proximity drawing that make it possible to significantly enlarge the families of representable graphs. Some challenging open problems on proximity drawings are listed in Section 4.5. Finally, Section 4.6 concludes the chapter by briefly pointing at two research directions in the areas of sensor networks and of robust geometric computing where proximity graphs and drawings have received some attention in the last few years.

4.2 Proximity Rules and Proximity Drawings

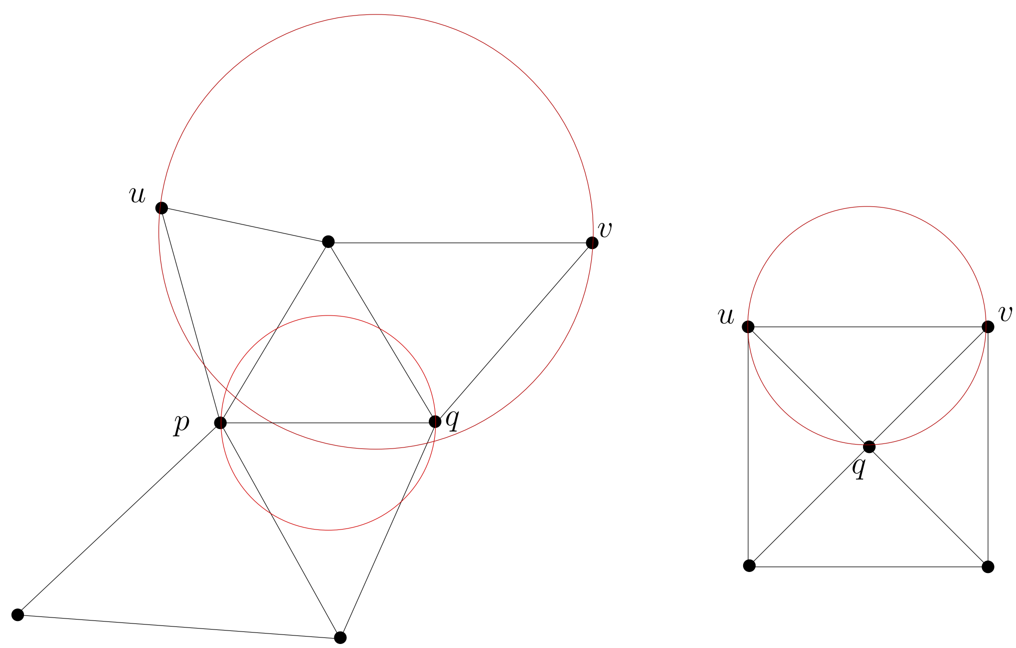
At a first, broad approximation, the definition of closeness in a proximity drawing can be either based on the concept of *proximity region* or based on a *global proximity* measure. In a proximity region based drawing two or more vertices are adjacent if and only if some suitably defined region that describes the neighborhood of these vertices contains at most k other vertices, for a given integer value $k \geq 0$. Global proximity, by contrast, gives rise to proximity drawings where the overall sum of the lengths of the edges in the drawing is minimized. In the remainder of this section, we recall some of the most common definitions of region-based and of global proximity rules and drawings; unless stated otherwise, P denotes the set of vertices of a straight-line drawing Γ of some graph G.

4.2.1 Proximity Region Based Drawings

Let R be a function that associates to every set S of $k \geq 2$ points in Euclidean d-space E^d a subset $R(S)$ of E^d; $R(S)$ is called the *proximity region* or *region of influence* of S.

Now consider a straight-line drawing Γ of G in which the vertices are drawn at a set of locations P. Drawing Γ is an (h,k)-*proximity drawing* of G if Γ results from the following procedure: For every set $S \subset P$ of h vertices, edges are drawn between all pairs of vertices in S if and only if the proximity region $R(S)$ contains at most k vertices from $P-S$. While the proximity region can be any subset of the space in question, usually the regions chosen are homeomorphic to an open or closed ball of dimension equal to that of the space. Such drawings are referred to as *open* or *closed* proximity drawings , respectively. Examples of open/closed (h,k)-proximity drawings follow.

The *Gabriel region* [GS69] of two vertices x and y is defined to be the closed sphere (in d dimensions) having the segment $\overline{xy}$ as diameter. A *Gabriel drawing* is a closed $(2,0)$-proximity drawing where the region of influence is the Gabriel region. Indeed, a Gabriel drawing of G is a straight-line drawing of G having the property that two vertices x and y of the drawing form an edge if and only if the Gabriel region of x and y does not contain any other vertex. Figure 4.1 (a) shows a Gabriel drawing of a planar triangulated graph; in the figure, vertices p and q are adjacent because their Gabriel region does not contain any other vertex, while vertices u and v are not adjacent because their Gabriel region contains vertex q. If one changes the definition of Gabriel region by saying that the sphere defined by the two vertices x and y is an open set, then the corresponding proximity region is termed a *modified Gabriel region* and the associated drawing is called a *modified Gabriel drawing*. Figure 4.1 (b) shows a modified Gabriel drawing of a wheel graph of five vertices; note that vertices u and v are adjacent because the modified Gabriel region is an open set and hence it does not contain vertex q. Note that the graph of Figure 4.1 (b) does not have a Gabriel drawing [Cim92].



(a) (b)

Figure 4.1 (a) A Gabriel drawing. (b) A modified Gabriel drawing of a graph that does not have a Gabriel drawing.

A *relative neighborhood drawing* of a graph G is an open $(2, 0)$-proximity drawing in which the region of influence of two points x and y is the intersection of the open disks of radius $d(x, y)$ centered at x and y. Thus, in a proximity drawing of G, x and y are adjacent if and only if there is no vertex whose distance to both x and y is less than the distance between x and y. The proximity region of x and y is called the *relative neighborhood region* or *lune* of x and y [Tou80].[1] A relative neighborhood drawing of a wheel graph consisting of a vertex of degree six adjacent to five vertices of degree three is depicted in Figure 4.2 (a): Since the relative neighborhood region is an open set, vertex w is not in the relative neighborhood region of vertices u and v and therefore the two vertices are adjacent.

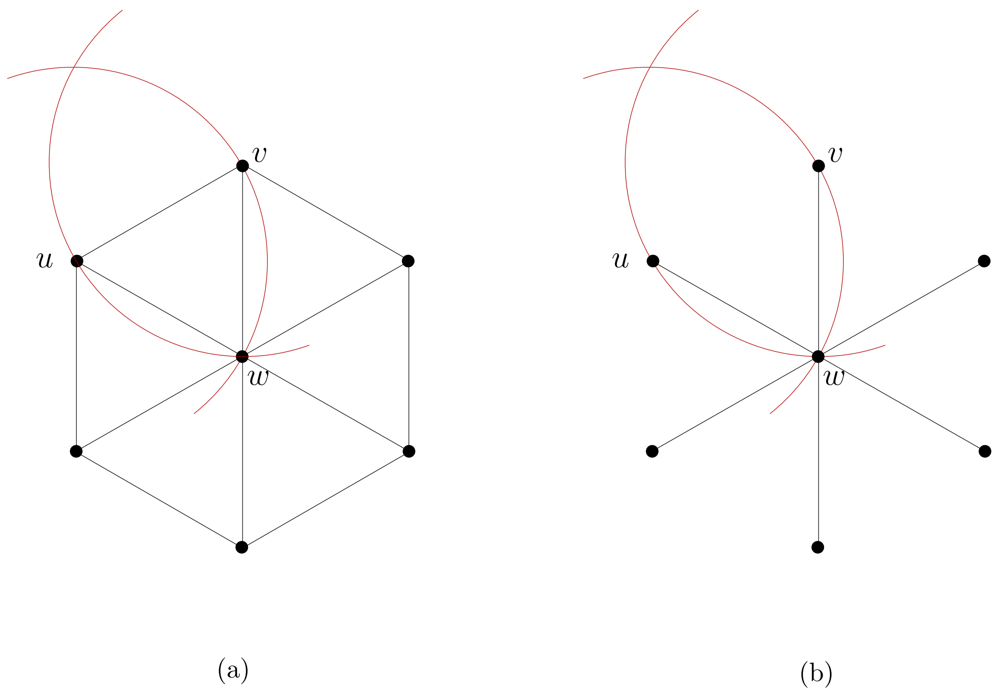


Figure 4.2 (a) A relative neighborhood drawing. (b) A relatively closest drawing of a tree that does not have a relative neighborhood drawing.

Variants of relative neighborhood graphs have also been studied. One example is the k-*relative neighborhood drawing* [CTL92] where the proximity constraint is relaxed by saying that two vertices are adjacent if and only if their lune contains at most k other vertices (for a given $k > 0$). As another example, if the relative neighborhood region is assumed to be a closed set, we have the so-called *relatively closest region* [Lan69] and *relatively closest drawing*. Figure 4.2 shows how proximity drawability differs for relative neighborhood drawings and relatively closest drawings. The drawing of Figure 4.2 (b) uses the same vertex set as the one of Figure 4.2 (a); however, the vertices u and v of Figure 4.2 (b) are

[1]While the term lune is commonly used in the computational geometry literature to denote the relative neighborhood region, it has to be recalled that in plane geometry the non-empty intersection of two disks of equal radius is called *symmetric lens*.

not adjacent because their relatively closest region, which is a closed set, contains vertex w. Note that a tree with one interior vertex and six leaves does not admit a relative neighborhood drawing [BLL96].

The Gabriel, modified Gabriel, relative neighborhood, and relatively closest drawings described above are all examples of members of a family of drawings called *β-drawings*. In 1985, Kirkpatrick and Radke [KR85, Rad88] introduced a family of closed $(2,0)$-proximity regions called *β-neighborhoods*, denoted by $R[x,y,\beta]$ and defined as follows (see also Figure 4.3):

1. For $\beta = 0$, $R[x,y,\beta]$ is the line segment $\overline{xy}$.
2. For $0 < \beta < 1$, $R[x,y,\beta]$ is the intersection of the two closed disks of radius $d(x,y)/(2\beta)$ passing through both x and y.
3. For $1 \leq \beta < \infty$, $R[x,y,\beta]$ is the intersection of the two closed disks of radius $\beta d(x,y)/2$ and centered on the line through x and y.
4. For $\beta = \infty$, $R[x,y,\beta]$ is the closed infinite strip perpendicular to the line segment $\overline{xy}$.

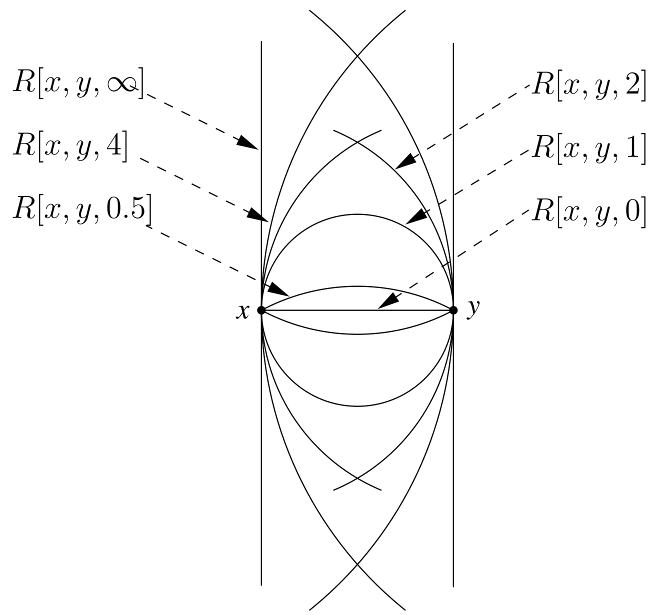


Figure 4.3 A set of $(2,0)$-proximity regions $R[x,y,\beta]$.

Obviously, one can also define the analogous regions $R(x,y,\beta)$ using open sets instead of closed sets ($R(x,y,0)$ is defined to be the empty set). The Gabriel, modified Gabriel, relative neighborhood and relatively closest drawings mentioned above are obtained from the β-regions $R[x,y,1]$, $R(x,y,1)$, $R(x,y,2)$ and $R[x,y,2]$, respectively. The *closed strip drawings* are β-drawings that use the region $R[x,y,\infty]$. Similarly, the *open strip drawings* are β-drawings that use the region $R(x,y,\infty)$. The regions defined above are also referred to as *lune-based* β-regions. In the same papers, Kirkpatrick and Radke [KR85, Rad88] also describe *circle-based* β-regions: for each $\beta \geq 1$, the region associated with two vertices x and y is the union of the two disks of radius $\beta d(x,y)/2$ passing through both x and y and centered on the line through them.

In the $(2, 0)$-proximity drawings described above, the proximity region chosen for a pair of vertices x, y is symmetric about the perpendicular bisector of the segment $\overline{xy}$. This guarantees a certain symmetry in the drawings produced. This symmetry, however, is not always desirable. Veltkamp [Vel92, Vel94, Vel95] introduced a family of proximity regions, called *γ-regions*, in which the proximity region may lack this symmetry and that generalize lune-based and circle-based β-regions. While Veltkamp takes advantage of this absence of symmetry in constructing object boundaries from a set of points in the context of visual processing and pattern recognition, the notion of γ-regions can be used from a graph drawing perspective to define *γ-drawings*. Another generalization of proximity drawings based on β-region are the so-called *empty region graphs*, recently introduced by Cardinal, Collette, and Langerman [CCL09]. An empty region graph is a proximity drawing where the proximity region of any pair of points u and v in the plane is a *template region*, that is a function mapping the pair u, v to a subset of the plane. In particular, the authors focus on the combinatorial properties of proximity graphs whose template regions are convex and symmetric.

Several $(2, 0)$-proximity regions can be seen as special cases of either β-regions or γ-regions; however, there are some well-known proximity regions defined in the literature that cannot be classified as members of some parameterized infinite family. Among those that have been investigated in the graph drawing context, we recall here the *rectangle of influence* [IS85] region, for which the proximity region associated with two points x and y is the axis-parallel rectangle determined by x and y. As in the case of β-drawings, one can use either open or closed rectangles; as with β-regions, the choice will determine which graphs can be drawn. A proximity drawing that uses the (open or closed) rectangle of influence region is called (open or closed) *rectangle of influence drawing*. In this type of drawing two vertices x and y are connected by an edge if and only if the (open or closed) rectangle of influence of x and y does not contain any other vertex. Figures 4.4 (a) and (b) show an open and a closed rectangle of influence drawing, respectively; the two drawings have the same vertex set.

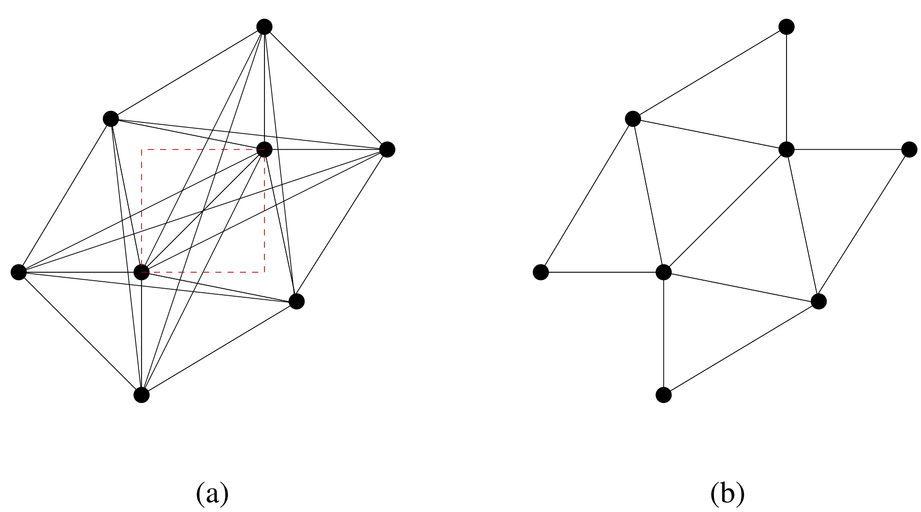


Figure 4.4 (a) An open rectangle of influence drawing; the dotted box represents the rectangle of influence of two vertices. (b) A closed rectangle of influence drawing.

The $(2, k)$-proximity drawing paradigm can also be used to produce drawings of directed graphs by associating with each *ordered* pair of points (x, y) a proximity region $R_{x,y}$. By allowing the region $R_{x,y}$ to be different from the region $R_{y,x}$, it is possible to produce drawings where the edge (x, y) is in the drawing, but not the edge (y, x). An early example of this is the *nearest neighbor drawing* (see, e.g, [PY92]), where each vertex $x \in P$ is connected to all vertices (or sometimes just one) of minimum distance from x. Although the nearest neighbor drawing is usually considered to be an undirected graph, the definition is inherently that of a directed graph. The proximity region $R_{x,y}$ in this case is the open disk of radius $d(x, y)$ centered at x.

Besides $(2, h)$-proximity drawings, there are many other meaningful and well-investigated families of proximity drawings. A *Delaunay drawing* [Del34] is an example of a closed $(3, 0)$-proximity drawing: here triplets of points in P are connected into triangles if and only if the closed disk they determine contains no other points of P. Delaunay drawings make sense for planar triangulated graphs (a Delaunay drawing is commonly called a *Delaunay triangulation* in the computational geometry literature). A *Delaunay drawing of order* h (usually called a *higher order Delaunay triangulation* in the computational geometry literature [GHvK02]) is a $(3, h)$-proximity drawing of a planar triangulated graph, where for every triplet of vertices connected into a triangle the closed disk through the triplet contains at most h other points of P (for a given integer $h \geq 0$).

Related to Delaunay triangulations is another well-known proximity graph, namely, the *Voronoi diagram* (see, e.g., [PS90]). A Voronoi diagram of a set of points P is the geometric dual of the Delaunay triangulation of P, i.e., it is the straight-line drawing whose edges are the perpendicular bisectors of the edges of the Delaunay triangulation and whose vertices are the intersection points of these perpendicular bisectors. Equivalently, the Voronoi diagram of P, for a given metric, is a subdivision of the plane into regions such that each region is associated with a distinct point p of P and it contains all points of the plane that are closer to p than to any other elements of P. We can therefore define a new type of proximity drawing: A *Voronoi drawing* [LM03] of a graph G is a straight-line drawing of G that is also the Voronoi diagram of some set of points (also called *sites*).

Figures 4.5 (a) and (b) show examples of Voronoi drawings in the Euclidean and in the Manhattan metric, respectively. In the figure, the white points are the sites; for display purposes, the edges of infinite length of the Voronoi diagram have been replaced by edges of finite length and endvertices have been added.

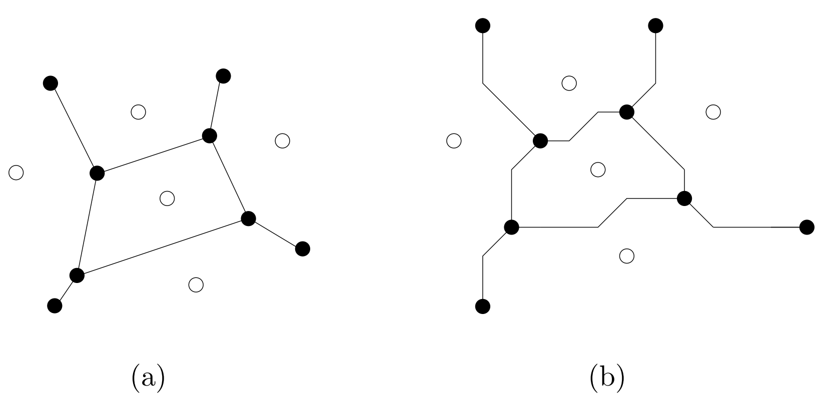


Figure 4.5 Voronoi drawings: (a) in the Euclidean metric and (b) in the Manhattan metric.

In our definition of (k, n)-proximity drawings, we have required that the sets S to which we associate proximity regions contain at least two points, since otherwise no edges can be formed. There is, however, a way in which proximity regions associated with single points can be used to create proximity drawings: pairs of points can be connected by an edge if the regions corresponding to the points intersect. We call such drawings *intersection drawings*. An example of such a proximity drawing would be a *sphere of influence drawing* of a graph. To produce this type of drawing, each point $x \in P$ has, as its proximity region, its *sphere of influence* [Tou88], namely, the disk centered at x of radius $r_x = \min\{d(x, y) : y \in P - \{x\}\}$. One can consider either open or closed sphere of influence drawings. An example of a sphere of influence graph is depicted in Figure 4.6; the drawing is valid for both the open and the closed model of proximity.

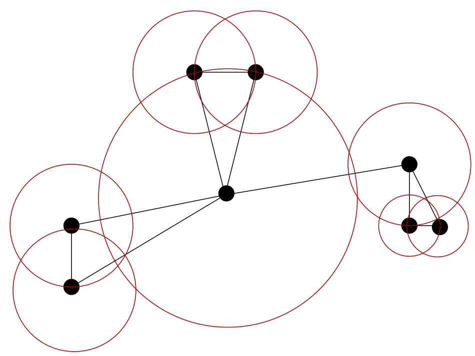


Figure 4.6 A sphere of influence drawing.

4.2.2 Global Proximity

Several graph drawing algorithms are designed to produce a representation of a graph that is as small as possible in some sense. For example, given a resolution rule (i.e., a minimum acceptable distance between any pair of graphic features in the drawing) one may want to optimize the area of the drawing or aim for minimum edge lengths. A proximity drawing that adopts a global measure of proximity is, in a sense, the smallest possible representation of a graph because it globally maximizes the closeness of adjacent vertices and the reciprocal distances of those pairs that are not adjacent.

The *weight* of a drawing of a graph is defined to be the sum of the lengths of the edges of the drawing. Frequently, drawings of graphs are required to satisfy some set of *aesthetic criteria* such as planarity or orthogonality. For a graph G, a set P of points in the plane, and a set $\mathcal{E}$ of aesthetic criteria, the *weight of G with respect to P*, denoted by $w_P(G)$, is defined as follows: $w_P(G)$ is the minimum taken over the weights of all drawings of G having P as the vertex set and satisfying $\mathcal{E}$; if no such drawing exists, then $w_P(G) = \infty$. Now let $\mathcal{C}$ be a class of graphs. A graph $G \in \mathcal{C}$ is *minimum weight drawable (for $\mathcal{C}$)* if there exists a set P of points such that $w_P(G)$ is finite and G minimizes $w_P()$ over all graphs in $\mathcal{C}$. Any drawing of G that achieves this minimum value is called a *minimum weight drawing of G with respect to P*. Two well-known examples of such drawings are given below (see, e.g., [PS90]).

A *minimum spanning tree* of a set P of points is a connected, straight-line drawing that has P as its vertex set and that minimizes the total edge length. So, letting $\mathcal{C}$ be the class of all trees, and letting $\mathcal{E}$ denote straight-line planar drawings, a tree G is minimum weight drawable for $\mathcal{C}$ if there exists a set P of points in the plane such that G minimizes $w_P()$ over all trees. This is equivalent to saying that G is isomorphic to a minimum spanning tree of P. Figure 4.7 shows a minimum weight drawing of a tree. A *minimum weight triangulation* of a set P is a triangulation of P having minimum total edge length. Letting $\mathcal{C}$ be the class of all planar triangulations, and letting $\mathcal{E}$ be as above, a planar triangulation G is minimum weight drawable for $\mathcal{C}$ if there exists a set P of points in the plane such that G is isomorphic to a minimum weight triangulation of P.

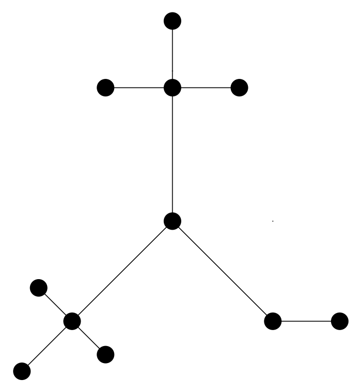


Figure 4.7 A minimum weight drawing of a tree.

We conclude this section by remarking that there are strong relations between global and region-based proximity rules. For example, it is well known that every minimum spanning tree on a set P of points is a subgraph of the Delaunay triangulation of P (see, e.g., [PS90]). Also, a significant research effort can be found in the computational geometry literature devoted to studying the relationships between $(2, k)$-proximity and minimum weight triangulations (see, e.g., [Kei94, WY01, CX01]).

4.3 Results

In this section, we survey some of the most relevant results on the proximity drawability problem for the types of proximity drawings described in the previous section.

4.3.1 Minimum Weight Drawings

If a graph admits a minimum weight drawing, it is called *minimum-weight drawable*; otherwise, it is called *minimum-weight forbidden*. As already mentioned above, most research on minimum weight drawings has focused on trees and on planar triangulated graphs.

The problem of testing whether a tree can be drawn as a Euclidean minimum spanning tree in the plane is essentially solved. Monma and Suri [MS92] proved that each tree with maximum vertex degree at most five can be drawn as a minimum spanning tree of some set of vertices by providing a linear-time (real RAM) algorithm. In the same paper, it is shown that every tree having at least one vertex with degree greater than six is minimum weight forbidden. As for trees having maximum degree equal to six, Eades and Whitesides [EW96b] showed that it is NP-hard to decide whether such trees can be drawn as minimum spanning trees.

One of the most challenging questions in the seminal paper by Monma and Suri [MS92] was about the area required by a minimum weight drawing of a tree. Namely, the construction by Monma and Suri used a grid of size $O(2^{n^2}) \times O(2^{n^2})$ and the authors conjectured an exponential lower bound for minimum weight drawings of trees with maximum vertex degree five (i.e., the existence of a tree T with n vertices such that any minimum weight drawing of T requires area at least $c^n \times c^n$ for some constant $c > 1$). This long-standing conjecture was only recently proved to be correct by Angelini et al. [ABC+11], who describe a tree T with n vertices having maximum degree five such that in any minimum weight drawing of T the ratio between the longest and the shortest edge is $2^{\Omega(n)}$, which implies that the drawing requires exponential area.

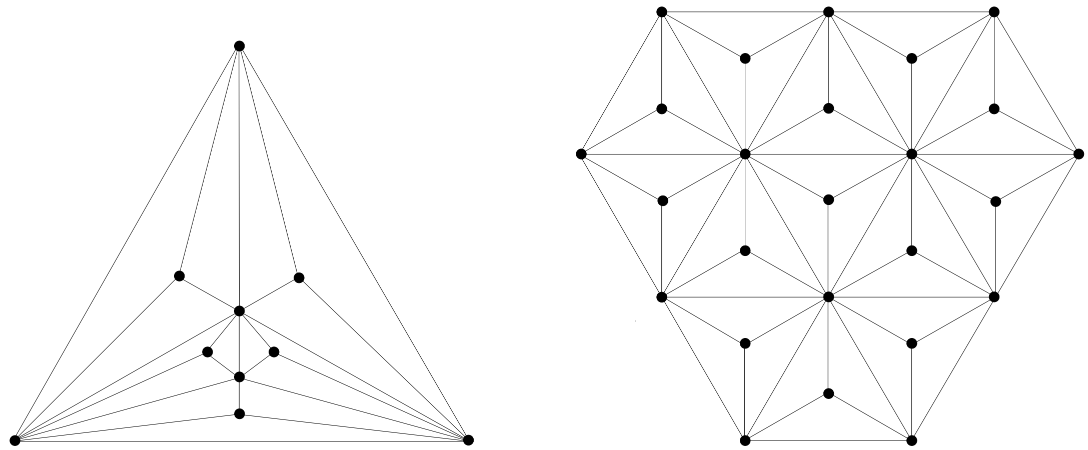
On the other hand, Frati and Kaufmann [FK11] proved that the exponential area lower bound of minimum weight drawings of trees does not hold for maximum vertex degree smaller than five. More precisely, let T be any tree with n vertices and maximum vertex degree four; Frati and Kaufmann show how to compute a minimum weight drawing of T with the following area upper bounds: (i) $O(n^{4.3})$ if T is a complete binary tree; (ii) $O(n^{11.3875})$ if T is an arbitrary binary tree; (iii) $O(n^{3.73})$ if T is a complete ternary tree; (iv) $O(n^{21.252})$ if T is an arbitrary ternary tree. The area bound for complete binary tree has been further reduced to $O(n^{3.8})$ by Di Giacomo et al. [DDLM12] (see also Section 4.4.3).

The 3-dimensional question about characterizing those trees that can be drawn as a Euclidean minimum spanning tree is not yet completely solved. In [LD95], it is shown that every tree having at least one vertex with degree greater than twelve is minimum weight forbidden in 3-dimensional space while all trees with vertex degree at most nine are drawable. King [Kin06] improved this last result by showing that all trees whose vertices have vertex degree at most ten can be realized as a Euclidean minimum spanning tree in 3-dimensional space. In general, the maximum vertex degree of a minimum weight drawable tree is bounded by the *kissing number*, i.e., by the maximum number of disjoint unit spheres that can be simultaneously tangent to a given unit sphere [RS95].

A significant research effort has also been devoted to drawing a planar triangulated graph G as a minimum weight triangulation of the points representing the vertices. However, the problem is still far from being solved. It may be worth recalling that, while computing a Euclidean minimum spanning tree of a set of points in the plane is solvable in polynomial time (see, e.g. [PS90]), the problem of computing a Euclidean minimum weight triangulation of a set of points in the plane is NP-hard [MR08].

In [LL96], it is shown that all maximal outerplanar triangulations are minimum-weight drawable, and a linear time (real RAM) drawing algorithm for constructing such a drawing is given. This naturally leads to investigation of the internal structure of minimum-weight drawable planar triangulated graphs. In [LL02] the authors examined the *endoskeleton*—or *skeleton*, for short—of planar triangulated graphs, that is, the subgraph induced by the internal vertices of the triangulation. They constructed skeletons that cannot appear in any minimum weight drawable triangulation and skeletons that guarantee minimum weight drawability. More precisely, the known results about of minimum weight drawable triangulations are as follows.

- In [LL02], the authors showed that any forest can be realized as the skeleton of some minimum weight triangulation. On the other end, Wang, Chin, and Yang [WCY00] gave examples of triangulations that do not admit a minimum weight drawing even if their skeleton is acyclic.
- In [LL02], it is also shown that any traingulation containing either the graph of Figure 4.8 (a) or the graph of Figure 4.8 (b) is not minimum weight drawable.



(a) (b)

Figure 4.8 Two examples of triangulations that cannot be drawn as minimum-weight triangulations.

Another contribution of [LL02] is to study the relationship between Delaunay drawability and minimum weight drawability. The authors described graphs that do not admit a Delaunay drawing but do have a minimum weight drawing. One such example is the minimum weight drawing of Figure 4.9: As explained in the next section,the depicted graph violates a necessary condition for Delaunay drawability (see also Figure 4.11 (b)).

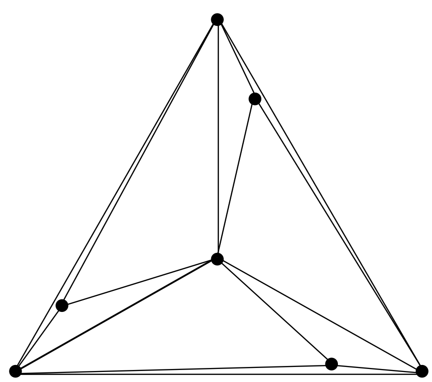


Figure 4.9 A minimum weight drawing of a Delaunay forbidden graph (see also Figure 4.11 (b).

4.3.2 Delaunay and Voronoi Drawings

The study of the combinatorial properties of Delaunay triangulations and of Voronoi diagrams (i.e., the Delaunay and the Voronoi drawability problems) has a long tradition in the computational geometry literature and is of particular interest because it is closely related to the design of topologically consistent algorithms for computing Delaunay/Voronoi diagrams in finite precision (see, e.g., [SI92, SH97, SIII00]).

The problem of characterizing which graphs admit Voronoi drawings has been studied in [LM03] both for the Euclidean and for the Manhattan metric. It is shown that every tree, independently of its maximum vertex degree, can be drawn as the Voronoi diagram of some set of points in the Euclidean metric. It is also proved that the maximum vertex degree of a Voronoi drawable tree in the Manhattan metric is at most five and that this bound is tight. Finally, the family of those binary trees that admit a Voronoi drawing in the Manhattan metric is characterized. Figure 4.10 shows examples of Voronoi drawings of trees in the Euclidean and in the Manhattan metric.

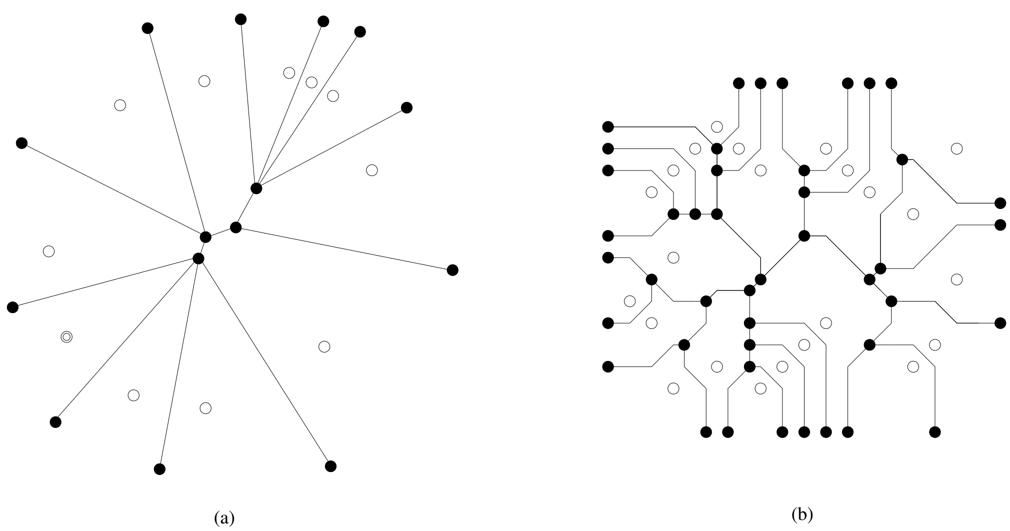


Figure 4.10 Two Voronoi drawings of trees: (a) a Euclidean Voronoi drawing and (b) a Manhattan Voronoi drawing. The white circles are sites and the black circles are vertices of the drawing. For display purposes, the edges of infinite length of the Voronoi diagrams have been replaced by edges of finite length and endvertices have been added.

An exact characterization of those graphs that admit Delaunay drawings remains a challenging open problem; however, some sufficient conditions and some necessary conditions for Delaunay drawability are known. Dillencourt [Dil90a] proved that if a graph G is maximal outerplanar, then G is Delaunay drawable. In a different paper, Dillencourt [Dil90b] studied the relationship between Delaunay drawability and 1-toughness. A graph G is *1-tough* if for any non-empty set S of vertices of G, the number of components obtained from G by removing the vertices of S and their incident edges is at most $|S|$. For example, the graph of Figure 4.11 (a) is not 1-tough because the removal of the four white vertices and of their incident edges results in a graph with five components. Dillencourt showed in [Dil90b] that every Delaunay drawable graph either (a) is 1-tough or (b) for any set S of vertices of G, the number of interior components obtained from G by removing the vertices of S and their incident edges is at most $|S| - 2$ (an interior component is a component that has no vertices in the outerface of G). This necessary condition is used by Dillencourt to construct examples of graphs that are nor Delaunay drawable. For example, neither graph of Figure 4.11 is Delaunay drawable: as already explained, the one of Figure 4.11 (a) violates the 1-toughness condition; the one of Figure 4.11 (b), violates the second necessary condition stated above because removing the four white vertices gives rise to three interior components. Another interesting property proved in [Dil90b] is that any Delaunay drawable graph has a perfect matching.

Based on the strict connection between the convex hull of a set of non-coplanar points on the surface of a sphere and a (2-dimensional) Delaunay triangulation [Bro79], the following equivalent definition of Delaunay drawable graphs was also given by Dillencourt [Dil96]: A planar triangulated graph G with triangular outerface is Delaunay drawable if and only if it is inscribable, i.e., it can be drawn in 3-dimensional space as the convex hull of a set of non-coplanar points on the surface of a sphere. If the outerface f of G is not triangulated, then G is Delaunay drawable if and only if the graph obtained from G by "stellating" f (i.e., by adding a vertex in f and connecting it to all vertices of f) is inscribable. Dillencourt and Smith [DS95] showed that every planar triangulated graph whose vertices all have degree three is inscribable (after having possibly stellated the outerface) and therefore Delaunay drawable. The same authors showed in [DS94] that any 4-connected planar graph is inscribable and that any triangulated graph with triangular outerface and without chords or non-facial triangles is Delaunay drawable.

The question whether Delaunay drawable graphs are Hamiltonian was posed by Mathieu [Mat87] and by O'Rourke [O'R87]. Examples of Delaunay drawable graphs that are not Hamiltonian can be found in papers by Dillencourt [Dil87, Dil89] and by Kantabutra [Kan83]. These examples suggested the question of the computational complexity of the Hamiltonicity of Delaunay drawable graphs. The question was answered by Dillencourt [Dil96], who proved that determining whether a Delaunay drawable graph is Hamiltonian is NP-complete. In the same paper it is also shown that there exist Delaunay drawable graphs that do not have a 2-factor (a 2-factor of a graph is a spanning collection of disjoint cycles). Finally, in the papers by Di Battista and Vismara [DV96], and by Sugihara and Hiroshima [SH97], the angles of Delaunay drawings were characterized.

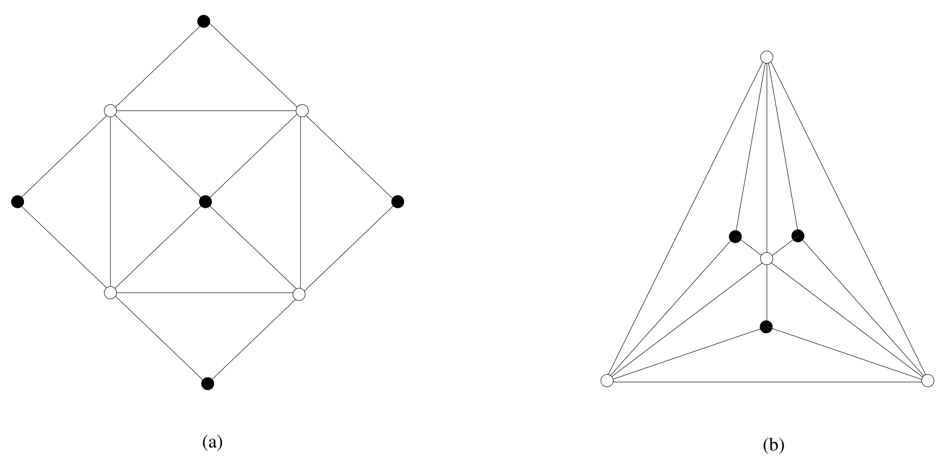


Figure 4.11 Two graphs that are not Delaunay drawable: (a) The graph is not 1-tough: removing the white vertices produces five components. (b) Removing the four white vertices produces three internal components.

4.3.3 Rectangle of Influence Drawings

The rectangle of influence drawability problem was first defined in [LLMW98], where both the case that the rectangle of influence is an open set and the case that it is a closed set are investigated. For both cases, characterization results are presented concerning cycles, wheels, trees, outerplanar graphs, and cliques. As already observed, the set of representable graphs can be quite different, depending on whether the open or the closed rectangle of influence is used to define the proximity drawing. For example, Figure 4.12 (a) shows a closed rectangle of influence drawing of a 4-cycle, which is not an open rectangle of influence drawable graph. Figure 4.12 (b) gives an open rectangle of influence drawing of K_5 (i.e., the complete graph on five vertices), which is not a closed rectangle of influence drawable graph.

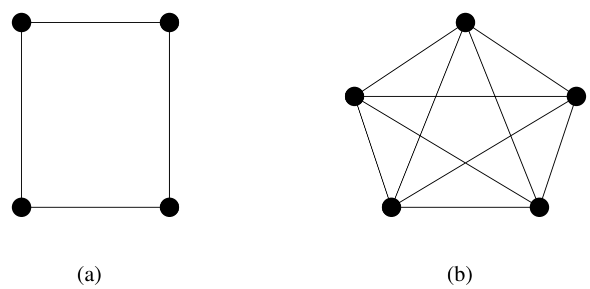


Figure 4.12 Examples of rectangle of influence drawings: (a) a closed rectangle of influence drawing of a 4-cycle; (b) an open rectangle of influence drawing of K_5. Note that a 4-cycle does not admit an open rectangle of influence drawings and that k_5 is not closed rectangle of influence drawable.

4.3.4 Nearest Neighbor Drawings

Paterson and Yao [PY92] started the investigation of the combinatorial properties of nearest neighbor graphs. Among other basic results, they proved that a nearest neighbor drawable tree cannot branch too much: if the depth of the tree is high, then the tree contains some long paths. More precisely, Paterson and Yao showed that if a tree of depth D is nearest neighbor drawable, then it can have at most $O(D^9)$ vertices. The upper bound was reduced to $O(D^6)$ by Eppstein [Epp92] and to $O(D^5)$ by Eppstein, Paterson, and Yao [EPY97]; this last upper bound is tight since Paterson and Yao [PY92] had shown the existence of nearest neighbor drawable graphs of depth D and $\Omega(D^5)$ vertices.

A precise characterization of nearest neighbor drawable graphs is still unknown. Eppstein, Paterson, and Yao [EPY97] conjectured that deciding whether a given graph is nearest neighbor drawable is hard. The truth of the conjecture was proved by Eades and Whitesides [EW96a] who show that it is NP-hard to determine whether a graph G is nearest neighbor drawable by using a mechanical device, called "logic engine," that simulates the well-known NP-complete problem NOT-ALL-EQUAL-3SATISFIABILITY [GJ79] and that provides a proof paradigm based on an approach first used by Bhatt and Cosmodakis [BC87]. Kitching and Whitesides [KW04] extend the technique to 3-dimensional space and, by using a "3-dimensional" logic engine, prove that the mutual nearest neighbor drawability problem is NP-hard in 3-dimensional space. It may be worth recalling that the logic engine paradigm can be used to prove the hardness of other graph drawing problems such as, for example, determining whether a graph is a subgraph of the hexagonal tiling or an induced subgraph of the square or hexagonal tilings [Epp09].

4.3.5 Sphere of Influence Drawings

Basic properties of sphere of influence drawable graphs are discussed by Harary et al. [HJLM93]. Harary et al. showed that if a graph G is open/closed sphere of influence drawable, an induced subgraph of G may not necessarily be open/closed sphere of influence drawable. This nonhereditary property greatly complicates the problem of characterizing sphere of influence drawable graphs. The conjecture of Harary et al. that K_9 does not admit an open sphere of influence drawing remains, to date, an open problem.

On the positive side, Jacobson, Lipman, and McMorris [JLM95] proved that if G is triangle-free and admits a sphere of influence drawing, then any subgraph of G is also drawable. Jacobson, Lipman, and McMorris exploited this result to characterize those trees that admit an open/closed sphere of influence drawing: A tree is open sphere of influence drawable if and only if it has a perfect matching; a tree is closed sphere of influence drawable if and only if it contains a $\{P_2, P_3\}$-factor (see, e.g., [Har69] for a definition of $\{P_2, P_3\}$-factor).

The number of edges of sphere of influence drawable graphs was independently studied by several researchers. Avis and Horton [AH82] proved that the number of edges of an open sphere of influence drawable graph cannot be larger than $29n$, where n is the number of vertices. An upper bound of $21n$ had also been already proven by Besicovitch [Bes45] in 1945, although he was not aware of the application of his result to the sphere of influence drawability problem. The bound of Besicovitch had later been improved by Reifenberg [Rei48] in 1948 and independently by Bateman and Erdös [BE51] in 1951, who showed an upper bound of $18n$ for the problem. Michael and Quint [MQ94b] had lowered the bound to $17.5n$. The best-known upper bound is due to Soss [Sos99a], who showed that any open/closed sphere of influence drawable graph can have at most $15n$ edges.

The study of sphere of influence drawings has also been extended to d-dimensional space and/or to different metrics (see, e.g., [GPS94, Sos99b, MQ99, MQ03]). The interested reader is also referred to the papers by Michael and Quint [MQ94a, MQ03] and to the work of Boyer, Lister, and Shader [BLS00] for more references and a list of open problems concerning the sphere of influence drawabality problem.

4.3.6 β-Drawings

Kirkpatrick and Radke [KR85, Rad88] defined the open and closed β-regions ($R(x, y, \beta)$, $R[x, y, \beta]$) discussed in the previous section. From the graph drawing perspective, the central problem is that of determining, for a given graph G, the values of β such that G admits a β-drawing. For example, for $\beta < 2$, only connected graphs admit β-drawings; for $\beta > 1$, only planar graphs do. As mentioned previously, the open and closed β-drawings include several well-studied proximity drawings, including Gabriel, Modified Gabriel, relative neighborhood, and relatively closest drawings; indeed, the Gabriel region is the closed β-region for $\beta = 1$, the modified Gabriel region is the open β-region for $\beta = 1$, the relative neighborhood region is the open β-region for $\beta = 2$, and the relatively closest region is the closed β-region for $\beta = 2$. Some papers about these types of drawings are described below.

Toussaint [Tou80] studied the relationship between the graphs produced by relative neighborhood drawings and other proximity drawings. He showed that the relative neighborhood drawing on a set P of points is a supergraph of every minimum spanning tree of P and a subgraph of the Delaunay triangulation of P. Agarwal and Matoušek [AM92] showed that the number of edges of an n-vertex graph that has a relative neighborhood drawing in 3-dimensional space is $O(n^{4/3})$. Chazelle, Edelsbrunner, Guibas, Hershberger, Seidel, and Sharir [CEG$^+$94] showed that the maximum number of edges of an n-vertex graph that has a Gabriel drawing in d-dimensional space ($d \geq 3$) is $\Omega(n^2)$. In [MS80], [Tou80], and [Lan69], the planarity of Gabriel drawable graphs, relative neighborhood graphs, and relatively closest drawable graphs were shown, respectively. Furthermore, in [Cim92] it was shown that a cycle with three vertices is not relatively closest drawable.

Particular attention has been devoted in the literature to β-drawings of trees. Matula and Sokal [MS80] gave a partial characterization of trees that admit Gabriel drawings. They proved that every tree with vertex degree at most three admits a Gabriel drawing, while no tree with vertex degree greater than six does. Urquhart [Urq83] gave the same two bounds on the vertex degree of relative neighborhood drawable trees. Cimikowski [Cim92] further extended the bounds to both modified Gabriel drawable and relatively closest drawable trees. Matula and Sokal [MS80] also conjectured that Gabriel drawable trees cannot have vertices of degree greater than four and cannot have two adjacent vertices of degree four.

The gaps left open in the above papers between the smallest and the largest vertex degree of a representable tree were the subject of a paper by Bose et al. [BLL96], who presented a complete characterization of those trees that admit Gabriel, Modified Gabriel, relative neighborhood, and relatively closest drawings. They showed that a tree admits a relative neighborhood and a relatively closest drawing if and only if its maximum vertex degree is at most five; also, a tree has a modified Gabriel drawing if and only if its maximum vertex degree is at most three. As for Gabriel drawability, they proved the truth of the conjecture by Matula and Sokal and characterized the family of representable trees by exhibiting families of forbidden subtrees and by showing that every tree that does not contain members of these families is Gabriel drawable. In the same paper, Bose et al. also presented linear-time algorithms to test whether a tree admits one of the above proximity drawings; it is shown that if such a drawing exists, one can be constructed in linear time in the real-RAM model.

As for other β-neighborhoods, Kirkpatrick and Radke [KR85] studied open strip drawable graphs (i.e., graphs that have β-drawings that use the open β-region $R(x, y, \infty)$) and showed that neither non-planar graphs nor triangulated planar graphs admit open strip drawings. A characterization of closed strip drawable graphs (i.e., proximity graphs that use the $R[x, y, \infty]$ region) can be found in the work by Bose et al. [BDLL95], where it is shown that a graph admits a closed strip drawing if and only if it is a binary forest other than one of the following: two non-adjacent vertices, a vertex and a non-adjacent edge, or two non-adjacent edges.

Bose et al. [BDLL95] also studied the proximity drawability of trees in the whole spectrum of β-proximity regions. Let $\mathcal{T}(\beta)$ ($\mathcal{T}[\beta]$) be the class of trees that have a proximity drawing where the proximity region is the open (closed) β-region and let $\mathcal{T}_k$ be the set of all finite trees of maximum vertex degree at most k. In [BDLL95], a complete characterization of $\mathcal{T}(\beta)$ for all β values such that $0 \leq \beta \leq \frac{1}{1-cos(\frac{2\pi}{5})} \simeq 1.45$ or such that $3.23 \simeq \frac{1}{\cos(\frac{2\pi}{5})} < \beta < \infty$ is given. Also, a complete characterization of $\mathcal{T}[\beta]$ for all β values such that $0 \leq \beta < \frac{1}{1-cos(\frac{2\pi}{5})}$ or such that $\frac{1}{\cos(\frac{2\pi}{5})} \leq \beta \leq \infty$ is presented. For all β values not in the above intervals, the authors give a partial characterization: They show that all trees in $\mathcal{T}_4$ and only trees in $\mathcal{T}_5$ belong to $\mathcal{T}(\beta)$ and $\mathcal{T}[\beta]$.

Table 4.1 summarizes the known results about families of trees that admit a β-drawing for different values of β in 2-dimensional space (for proofs and detailed description of recognition and drawing algorithms, see [BDLL95, BLL96]). In the table, $\overline{\mathcal{T}}$ denotes the family of trees that have at least two adjacent vertices of degree three and $\overline{\overline{\mathcal{T}}}$ denotes the family of "forbidden" graphs defined in [BLL96]. Figure 4.13 shows a β-drawing of a tree with all non-leaf vertices having degree four; the drawing is computed with the technique of [BDLL95] and assumes the value $\beta = 4$.

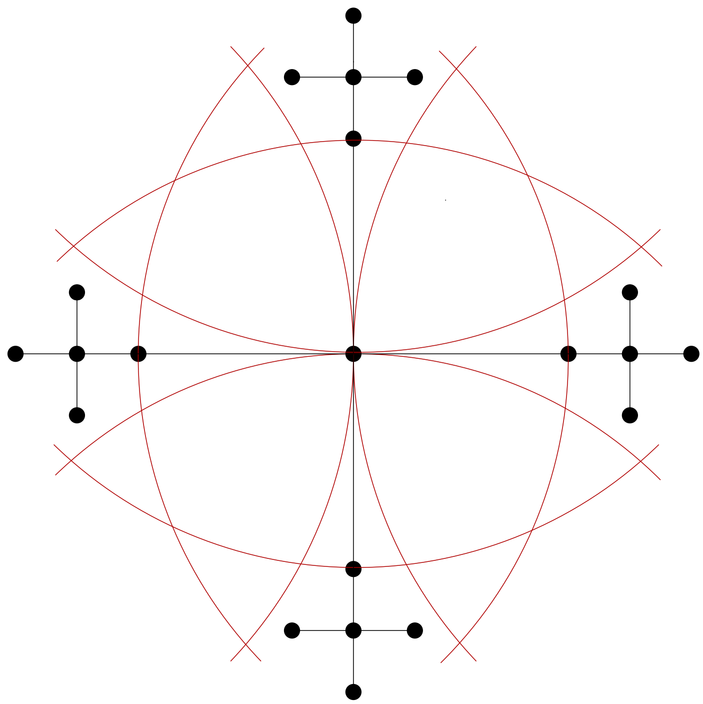


Figure 4.13 A β-drawing of a tree for $\beta = 4$ computed with the technique of [BDLL95].

The study of β-drawings of trees was also extended to 3-dimensional space. The definition of β-region recalled in the previous section can be straightforwardly extended to 3-dimensional space by considering open and closed 3-dimensional spheres instead of open and closed 2-dimensional spheres. In [LD95] it is shown that by using the third dimension the class of β-drawable trees becomes larger in many cases. For example, all trees having maximum vertex degree at most 4 are Gabriel drawable in 3-dimensional space, while this is not the case in the plane (see also Row 5 of Table 4.1); for $\beta = 2$ every tree having maximum vertex degree at most nine is drawable. The known results on β-drawability of trees in 3-dimensional space are summarized in Table 4.2, where the same notation of Table 4.1 is adopted; in the table, K_1 and K_2 denote the tree consisting of a single vertex and of a single edge, respectively.

Returning now to β-drawings in 2-dimensional space, the study of the β-drawability problem was extended from trees to outerplanar graphs by Lubiw and Sleumer [LS93], who showed that all maximal outerplanar graphs admit both a relative neighborhood drawing and a Gabriel drawing. They also proved that every biconnected outerplanar graph admits a relative neighborhood drawing. Lubiw and Sleumer also conjectured that any biconnected outerplanar graph admits a Gabriel drawing. This conjecture was settled in the affirmative in [LL97], where it is proved that indeed every biconnected outerplanar graph admits a β-drawing for all values of β such that $1 \leq \beta \leq 2$. In the same paper, the investigation was

	value of β	$\mathcal{T}(\beta)$	$\mathcal{T}[\beta]$
1	$\beta = 0$	$\mathcal{T}(\beta) = \{K_1, K_2\}$	$\mathcal{T}[\beta] = \mathcal{T}_2$
2	$0 < \beta < \frac{\sqrt{3}}{2}$	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_2$
3	$\beta = \frac{\sqrt{3}}{2}$	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_3 - \overline{\mathcal{T}}$
4	$\frac{\sqrt{3}}{2} < \beta < 1$	$\mathcal{T}(\beta) = \mathcal{T}_3$	$\mathcal{T}[\beta] = \mathcal{T}_3$
5	$\beta = 1$	$\mathcal{T}[\beta] = \mathcal{T}_4 - \overline{\overline{\mathcal{T}}}$	$\mathcal{T}[\beta] = \mathcal{T}_4 - \overline{\overline{\mathcal{T}}}$
6	$1 < \beta < \frac{1}{1-\cos(\frac{2\pi}{5})}$	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_4$
7	$\beta = \frac{1}{1-\cos(\frac{2\pi}{5})}$	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}_4 \subset T[\beta] \subset \mathcal{T}_5$
8	$\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subseteq \mathcal{T}_5$	$\mathcal{T}_4 \subset \mathcal{T}[\beta] \subseteq \mathcal{T}_5$
9	$\beta = 2$	$\mathcal{T}[\beta] = \mathcal{T}_5$	$\mathcal{T}[\beta] = \mathcal{T}_5$
10	$2 < \beta < \frac{1}{\cos(\frac{2\pi}{5})}$	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subseteq \mathcal{T}_5$	$\mathcal{T}_4 \subset T[\beta] \subseteq \mathcal{T}_5$
11	$\beta = \frac{1}{\cos(\frac{2\pi}{5})}$	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subset \mathcal{T}_5$	$\mathcal{T}[\beta] = \mathcal{T}_4$
12	$\frac{1}{\cos(\frac{2\pi}{5})} < \beta < \infty$	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_4$
13	$\beta = \infty$	$\mathcal{T}_3 \subset \mathcal{T}(\beta) \subset \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_3$

Table 4.1 β-drawability of trees for $0 \leq \beta \leq \infty$, 2-dimensional space.

extended to simply connected outerplanar graphs (notice that the family of these graphs includes trees as a special case); the authors show an upper bound on the number of biconnected components sharing a cutvertex in a β-drawable graph, for all possible values of β, which gives rise to partial characterization of the families of representable outerplanar graphs.

Table 4.3 summarizes the characterization results about the β-drawability of outerplanar graphs that can be found in [LS93, LL97]. All other entries describe results from this paper. $\mathcal{CO}$, $\mathcal{BO}$, and $\mathcal{MO}$ are the set of all connected outerplanar, biconnected outerplanar, and maximal outerplanar graphs, respectively. $\mathcal{G}_{\mathcal{CO}}(\beta)$, $\mathcal{G}_{\mathcal{BO}}(\beta)$, and $\mathcal{G}_{\mathcal{MO}}(\beta)$ are the classes of connected outerplanar, biconnected outerplanar, and maximal outerplanar (β)-drawable graphs, respectively. Similarly, $\mathcal{G}_{\mathcal{CO}}[\beta]$, $\mathcal{G}_{\mathcal{BO}}[\beta]$, and $\mathcal{G}_{\mathcal{MO}}[\beta]$ are the classes of connected outerplanar, biconnected outerplanar, and maximal outerplanar $[\beta]$-drawable graphs, respectively. $\mathcal{G}_k$ denotes the class of graphs such that the number of biconnected components sharing a cut-vertex is at most k.

Little is known about the β-drawability of graphs that are not outerplanar. Irfan and Rahman [IR07] gave a sufficient condition for the β-drawability of 2-outerplanar graphs for values of β in the interval $1 < \beta < 2$; they also described examples of 2-outerplanar graphs that do not admit a β-drawing for $1 < \beta < 2$. In the same paper, Irfan and Rahman described and $O(n^2)$-time algorithm to test their sufficient condition on a given 2-outerplanar graph with n vertices. The time complexity of this test was later improved to $O(n)$ by Samee, Irfan, and Rahman [SIR08].

4.4 Variations of Proximity Drawings

Some generalizations and relaxations of proximity drawings have been described in the literature. This section recalls three of them.

	β	$\mathcal{T}(\beta)$ 3-D	$\mathcal{T}[\beta]$ 3-D
1	$\beta = 0$	$\mathcal{T}(\beta) = \{K_1, K_2\}$	$\mathcal{T}[\beta] = \mathcal{T}_2$
2	$0 < \beta < \frac{2}{3}$	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_2$
3	$\beta = \frac{1}{2\sin^2(\frac{\pi}{3})} = \frac{2}{3}$	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_3 - \mathcal{T}'$
4	$\frac{2}{3} < \beta < \frac{1}{2\sin^2(\arcsin\sqrt{\frac{2}{3}})}$	$\mathcal{T}(\beta) = \mathcal{T}_3$	$\mathcal{T}[\beta] = \mathcal{T}_3$
5	$\beta = 0.75$	$\mathcal{T}(\beta) = \mathcal{T}_3$	$\mathcal{T}_3 \subset \mathcal{T}[\beta] \subseteq \mathcal{T}_4$
6	$\frac{3}{4} < \beta < \frac{1}{2\sin^2(\frac{\pi}{4})} = 1$	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_4$
7	$\beta = 1$	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}_4 \subset \mathcal{T}[\beta] \subset \mathcal{T}_6$
8	$1 < \beta < \frac{1}{2\sin^2(\frac{7\pi}{30})}$	$\mathcal{T}(\beta) = \mathcal{T}_6$	$\mathcal{T}[\beta] = \mathcal{T}_6$
9	$\frac{1}{2\sin^2(\frac{7\pi}{15})} \leq \beta < \frac{1}{2\sin^2(\frac{13\pi}{60})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_7$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_7$
10	$\frac{1}{2\sin^2(\frac{13\pi}{60})} \leq \beta < \frac{1}{2\sin^2(\frac{37\pi}{180})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_8$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_8$
11	$\frac{1}{2\sin^2(\frac{37\pi}{180})} \leq \beta < \frac{1}{2\sin^2(\frac{\pi}{5})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_9$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_9$
12	$\beta = \frac{1}{2\sin^2(\frac{\pi}{5})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_9$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_9$
13	$\frac{1}{2\sin^2(\frac{\pi}{5})} < \beta \leq \frac{1}{2\sin^2(\frac{7\pi}{36})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_9$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_9$
14	$\frac{1}{2\sin^2(\frac{7\pi}{36})} < \beta \leq \frac{1}{2\sin^2(\frac{67\pi}{360})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{10}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{10}$
15	$\frac{1}{2\sin^2(\frac{67\pi}{360})} < \beta \leq \frac{1}{2\sin^2(\frac{16\pi}{90})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{11}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{11}$
16	$\frac{1}{2\sin^2(\frac{16\pi}{90})} < \beta \leq \frac{1}{2\sin^2(\frac{61\pi}{360})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{12}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{12}$
17	$\frac{1}{2\sin^2(\frac{61\pi}{360})} < \beta < \frac{1}{2\sin^2(\frac{\pi}{6})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{13}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{13}$
18	$\beta = 2$	$\mathcal{T}_9 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{13}$	$\mathcal{T}_9 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{13}$
19	$2 < \beta < \frac{1}{\cos(\frac{61\pi}{180})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{13}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{13}$
20	$\frac{1}{\cos(\frac{61\pi}{180})} \leq \beta < \frac{1}{\cos(\frac{16\pi}{45})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{12}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{12}$
21	$\frac{1}{\cos(\frac{16\pi}{45})} \leq \beta < \frac{1}{\cos(\frac{67\pi}{180})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{11}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{11}$
22	$\frac{1}{\cos(\frac{67\pi}{180})} \leq \beta < \frac{1}{\cos(\frac{7\pi}{18})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_{10}$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_{10}$
23	$\frac{1}{\cos(\frac{7\pi}{18})} \leq \beta < \frac{1}{\cos(\frac{2\pi}{5})}$	$\mathcal{T}_7 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_9$	$\mathcal{T}_7 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_9$
24	$\beta = \frac{1}{\cos(\frac{2\pi}{5})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_9$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_9$
25	$\frac{1}{\cos(\frac{2\pi}{5})} < \beta < \frac{1}{\cos(\frac{37\pi}{90})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_9$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_9$
26	$\frac{1}{\cos(\frac{37\pi}{90})} \leq \beta < \frac{1}{\cos(\frac{13\pi}{30})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_8$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_8$
27	$\frac{1}{\cos(\frac{13\pi}{30})} \leq \beta < \frac{1}{\cos(\frac{7\pi}{15})}$	$\mathcal{T}_6 \subseteq \mathcal{T}(\beta) \subseteq \mathcal{T}_7$	$\mathcal{T}_6 \subseteq \mathcal{T}[\beta] \subseteq \mathcal{T}_7$
28	$\frac{1}{\cos(\frac{7\pi}{15})} < \beta < \infty$	$\mathcal{T}(\beta) = \mathcal{T}_6$	$\mathcal{T}[\beta] = \mathcal{T}_6$
29	$\beta = \infty$	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subset \mathcal{T}_6$	$\mathcal{T}[\beta] = \mathcal{T}_4$

Table 4.2 β-drawability of trees for $0 \leq \beta \leq \infty$, 3-dimensional space.

4.4.1 Witness Proximity Drawings

Witness proximity has been introduced and studied in a series of papers by Aronov, Dulieu, and Hurtado [ADH, ADH11a, ADH11b]. These papers study both the computational geometry problem of computing witness proximity graphs on a given point set and the graph drawing question of defining a set of points whose witness proximity drawing represents a given combinatorial graph. We recall here only those results relative to the proximity drawability problem.

Witness proximity drawings are region of influence based proximity drawings where the adjacency between pairs of vertices depends on whether the proximity region of these vertices contains or does not contain a point form a second set, called the *witness points.*

	β	Connected	Biconnected	Maximal
1	$\beta = 1$	$\mathcal{G}_2 \not\subseteq \mathcal{G}_{\mathcal{CO}}[1] \subset \mathcal{G}_4$	$\mathcal{G}_{\mathcal{BO}}[1] = \{\mathcal{BO}\}$	$\mathcal{G}_{\mathcal{MO}}[1] = \{\mathcal{MO}\}$
2	$1 < \beta < \frac{1}{1-\cos(\frac{2\pi}{5})}$	$\mathcal{G}_{\mathcal{CO}}(\beta) \subset \mathcal{G}_4$ $\mathcal{G}_{\mathcal{CO}}[\beta] \subset \mathcal{G}_4$	$\mathcal{G}_{\mathcal{BO}}(\beta) = \{\mathcal{BO}\}$ $\mathcal{G}_{\mathcal{BO}}[\beta] = \{\mathcal{BO}\}$	$\mathcal{G}_{\mathcal{MO}}(\beta) = \{\mathcal{MO}\}$ $\mathcal{G}_{\mathcal{MO}}[\beta] = \{\mathcal{MO}\}$
3	$\beta = \frac{1}{1-\cos(\frac{2\pi}{5})}$	$\mathcal{G}_{\mathcal{CO}}(\beta) \subset \mathcal{G}_4$ $\mathcal{G}_4 \not\subseteq \mathcal{G}_{\mathcal{CO}}[\beta] \subset \mathcal{G}_5$	$\mathcal{G}_{\mathcal{BO}}(\beta) = \{\mathcal{BO}\}$ $\mathcal{G}_{\mathcal{BO}}[\beta] = \{\mathcal{BO}\}$	$\mathcal{G}_{\mathcal{MO}}(\beta) = \{\mathcal{MO}\}$ $\mathcal{G}_{\mathcal{MO}}[\beta] = \{\mathcal{MO}\}$
4	$\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$	$\mathcal{G}_4 \not\subseteq \mathcal{G}_{\mathcal{CO}}(\beta) \subset \mathcal{G}_5$ $\mathcal{G}_4 \not\subseteq \mathcal{G}_{\mathcal{CO}}[\beta] \subset \mathcal{G}_5$	$\mathcal{G}_{\mathcal{BO}}(\beta) = \{\mathcal{BO}\}$ $\mathcal{G}_{\mathcal{BO}}[\beta] = \{\mathcal{BO}\}$	$\mathcal{G}_{\mathcal{MO}}(\beta) = \{\mathcal{MO}\}$ $\mathcal{G}_{\mathcal{MO}}[\beta] = \{\mathcal{MO}\}$
5	$\beta = 2$	$\mathcal{G}_4 \not\subseteq \mathcal{G}_{\mathcal{CO}}(2) \subset \mathcal{G}_5$	$\mathcal{G}_{\mathcal{BO}}(2) = \{\mathcal{BO}\}$	$\mathcal{G}_{\mathcal{MO}}(2) = \{\mathcal{MO}\}$

Table 4.3 β-drawability of outerplanar graphs for $1 \leq \beta \leq 2$, 2-dimensional space.

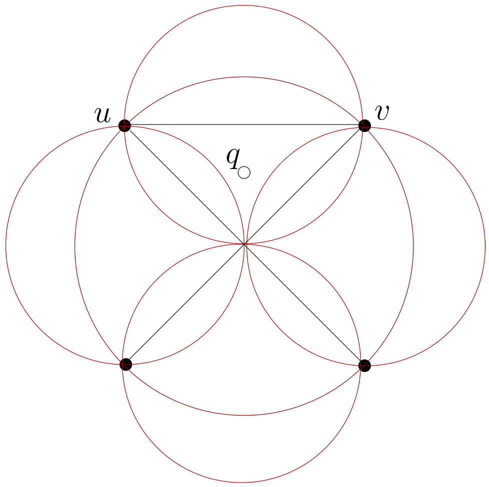
Therefore, in a witness proximity drawing, we look at a set of points that represent the vertices and at a set of points that play the role of the witnesses. The existence/absence of an edge in the drawing depends on the location of the witness points (the set of witness points and the set of points representing the vertices of the graph in drawing may not coincide).

In a *positive witness proximity drawing* Γ, two vertices (x, y) are adjacent if and only if the proximity region of x and y contains at least one vertex that belongs to the set of witness points. In a *negative witness proximity drawing*, x and y are adjacent if and only if their region of influence does not contain any of the witness points (it may however contain other vertices of the graph that are not witnesses). It is worth noticing that the definition of witness proximity drawing includes the notion of $(h, 0)$-proximity drawing as a special case: A negative proximity drawing where the set of witness points coincides with the vertex set is in fact an $(h, 0)$-proximity drawing.

The computation of a witness proximity drawing requires to define the set of points representing the vertices and the set of witness points. For example, Figure 4.14 (a) shows a positive witness Gabriel drawing and Figure 4.14 (b) a negative witness Gabriel drawing; the two drawings have the same witness point q and the same set of vertices. In the figures, the Gabriel disk of u and v contains the witness point q, which makes u and v adjacent in the positive witness Gabriel drawing.

In [ADH11a], Aronov, Dulieu, and Hurtado studied *witness Delaunay drawings*. More specifically, they consider negative witness Delaunay drawings, which are proximity drawings where two vertices x and y are adjacent if and only if there exists an open disk whose boundary passes through x and y and does not contain any point of the witness set. It is proved that every tree admits a negative witness Delaunay drawing for suitable set of witness points and that the drawing can be computed in linear time, adopting the real RAM model of computation. As for forbidden graphs, it is proved that non-planar bipartite graphs never admit a negative witness Delaunay drawing. In the same paper, positive witness Delaunay drawings in the L_∞ metric are studied. These drawings, also called *square drawings*, are such that two vertices x and y are adjacent if and only if there exists an axis-aligned square whose boundary passes through x and y and that contains at least one witness point. It is proved in [ADH11a] that a graph admits a square drawing if and only if it is a permutation graph and that a square drawing of a permutation graph can be computed by using at most one witness point.

The witness generalization of Gabriel drawings is studied in [ADH]. The paper describes both graphs that do not admit a negative witness Gabriel drawing and graphs that are negative witness Gabriel drawable. It is proved that all graphs containing $K_{3,3,3,3}$ as an



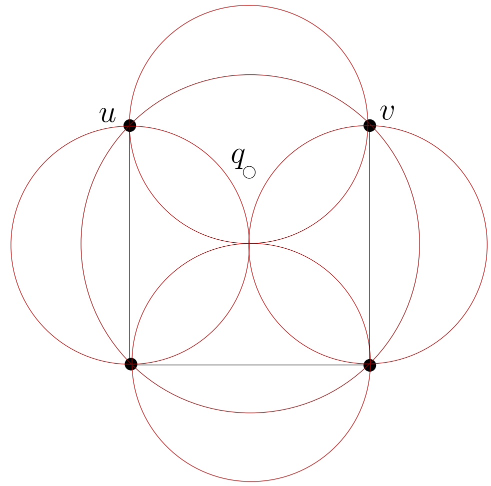


Figure 4.14 Two witness Gabriel drawings where q is the witness point: (a) Positive witness Gabriel drawing. (b) Negative witness Gabriel drawing.

induced subgraph do not have a negative witness Gabriel drawing. It is also proved that all trees are negative witness Gabriel drawable.

Positive witness rectangle of influence drawings are explored in [ADH11b]. In this paper, Aronov, Dulieu, and Hurtado show that a tree admits a positive witness rectangle of influence drawing if and only if it has no three independent edges. The paper also gives necessary conditions for positive witness rectangle of influence drawability of general graphs. Namely, a graph that has a positive witness rectangle of influence drawing has at most two non-trivial connected components (a connected component is non-trivial if its number of vertices is larger than one). If the graph has exactly two components, then each component has diameter three; if the graph has one component, it has diameter six. Finally, a characterization of the positive witness rectangle of influence drawable graphs having exactly two non-trivial components is given: A graph belongs to this family if and only if it is a disjoint union of zero or more isolated vertices and two co-interval graphs.

4.4.2 Weak Proximity Drawings

We recall that, a $(2, 0)$-proximity drawing Γ is a straight-line drawing such that: (i) for each edge (x, y) of Γ, the proximity region of x and y does not contain any other vertex, and (ii) for each pair of non-adjacent vertices x, y of Γ, the proximity region of x and y contains at least one other vertex. In this section, we shall call such drawings *strong* proximity drawings.

A relaxation of strong proximity drawings, called *weak proximity drawings*, was first introduced and studied in [DLW06]. A weak proximity drawing of a graph G is one that ignores requirement (ii). In other words, if x, y is *not* an edge of the graph, then no requirement is placed on the proximity region of x and y in the weak drawing. For example, Figure 4.15 (a) shows a weak proximity drawing of a tree. Here, the proximity region of any two points x and y is the disk having x and y as antipodal points. Note that the drawing is not a strong drawing, as no edges between neighbors of the degree six vertex are included.

The strong proximity drawing with the same proximity region and on the same set of points is shown in Figure 4.15 (b).

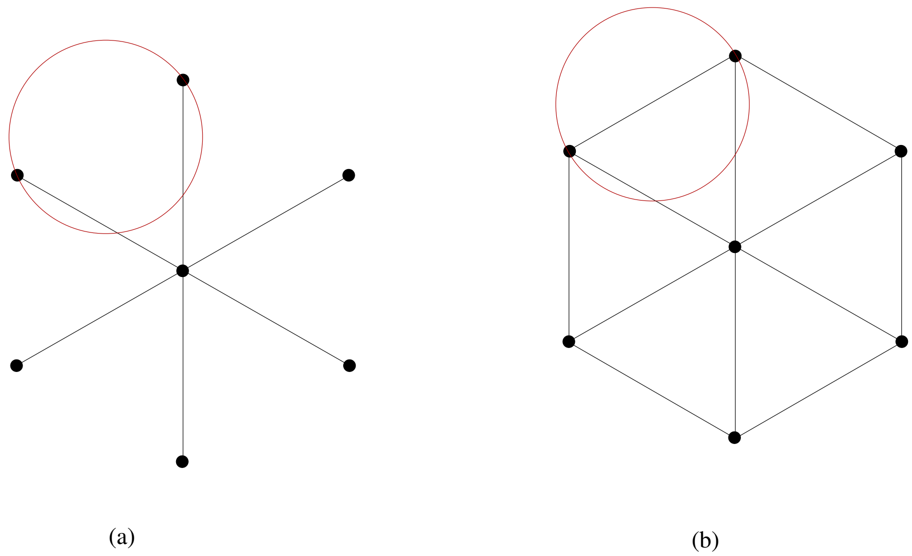


Figure 4.15 (a) A weak proximity drawing and (b) a strong proximity drawing.

For purposes of graph visualization, there are several reasons for studying weak proximity drawings. We summarize the ones that, in our opinion, are the most relevant.

- Strong proximity drawability may appear too restrictive for graph drawing purposes. By relaxing (ii), a graph G can no longer be reconstructed from the locations of its vertices in a weak drawing; however, many graphs that do not admit strong drawings can be drawn weakly. For example, a tree that has a vertex of degree greater than five has no strong 2-dimensional β-drawing for any β (see also Table 4.1). Thus, the drawing in Figure 4.15 (a) illustrates a graph that is *weak* but not *strong* drawable for the Gabriel region.
- A visibility drawing of a graph is a drawing such that (e.g., see [DETT99, KW01]) vertices are mapped to horizontal segments and edges are mapped to vertical segments that intersect only adjacent vertex segments. Of course, a necessary condition for drawing an edge is that the vertex segments corresponding to its end-vertices are visible in the vertical direction. If this condition is also sufficient, then we have a *strong visibility drawing*; otherwise, we have a *weak visibility drawing*. In the field of visibility drawing, the coordinated study of both strong and weak types of drawings led to deep and practical results.
- Weak proximity can be considered as an "edge-vertex resolution rule" in the sense that a vertex cannot enter the region of influence of an edge. Thus, the study of weak proximity can contribute to the body of drawing strategies that adopt a resolution rule (e.g., see [DETT99, KW01]).
- The weak proximity model may well be sufficient for many drawing applications, particularly ones that do not require recovery of the graph solely from the positions of its vertices. For example, weak proximity drawings have been receiving increasing interest for their applications to wireless network design, where dis-

tributed topology control can be based on proximity structures constructed from given geometric graphs by deleting those edges that do not satisfy a given proximity rule. The resulting graph is a weak proximity drawing because its edges satisfy the given proximity rule, whereas pairs of non-adjacent vertices may or may not contain other vertices in their proximity region. Papers devoted to the study of weak proximity graphs defined in the context of sensor networks include [CWL02, LCWW03, PS04, KL10]. See also Section 4.5 for more discussion and some other references about proximity and wireless ad-hoc networks.

In particular the research in [DLW06] focused on 2-dimensional weak β-drawing; the following results were proved.

General graphs: Any graph G is weak β-drawable for all β in the range 0 to some upper bound that is a function either of the number of vertices or of the maximum vertex degree of G.

Planar graphs: For any value of β such that $1 < \beta \leq \infty$, strong and weak β-drawings of triangulated planar graphs coincide. It was also shown how to interpret any straight-line drawing algorithm for planar triangulated graphs as an algorithm for constructing weak proximity drawings.

Trees: An algorithm was presented to draw any tree as a weak β-drawing for any value of β less than two. It was shown that for $2 \leq \beta < \infty$, the weak and the strong proximity models give rise approximately to the same class of 2-dimensional β-drawable trees. Finally, the NP-hardness of deciding whether a tree has a weak proximity drawing for $\beta = \infty$, where the region of influence is an open strip, was proved.

Table 4.4 schematically compares the known results on weak β-drawability against those on strong β-drawability for trees. Each row corresponds to a different interval of β and reports the maximum vertex degree k that a tree can have to admit a strong or weak β-drawing for some values of β in the interval. Of particular interest is the value $\beta = 2$, where remarkable differences in the drawable trees can be noticed, depending on whether the region of influence is an open set (in which case it coincides with the relative neighborhood region) or a closed set (in which case it coincides with the relatively closest region).

	value of β	strong β-drawability	weak β-drawability
1	$0 \leq \beta < 2$	$k \leq 5$	$k = \infty$
2	$\beta = 2$	$k = 5$	$k = \infty$ (w-(β)-draw.), $k = 5$ (w-[β]-draw.)
3	$2 < \beta \leq \infty$	$k \leq 5$	$k \leq 5$

Table 4.4 Comparing weak β-drawability of trees vs. strong β-drawability of trees. In the table, w-(β)-drawable means that the tree has a weak β-drawing where the β-region is an open set and w-[β]-drawable means that the tree has a weak β-drawing where the β-region is a closed set.

The advantage of using a weak model of proximity was also highlighted in [LL97], where it was proved that, in contrast with the results in Table 4.3, every connected outerplanar graph admits a weak Gabriel drawing, a weak relative neighborhood drawing, and a weak β-drawing for any given β such that $1 < \beta < 2$.

A comparison of strong and weak β-drawings in terms of area requirement can be found in [LTTV97] and in the work by Penna and Vocca [PV04]. Penna and Vocca [PV04] extended the study of weak proximity β-drawings to 3-dimensional space and proved several polynomial area/volume bounds for families of graphs for which a strong proximity drawing is either not admitted or requires exponential area. In general however, there exist families of graphs for which a Gabriel drawing in 2-dimensional space requires exponential area both for the strong and the weak model of proximity [LTTV97].

Weak nearest neighbor graphs were studied by Eades and Whitesides [EW96a], who showed that the problem of deciding whether a graph admits a weak nearest neighbor drawing is NP-hard. Thus, nearest neighbor drawability is NP-hard both in the weak and in the strong proximity model (see also Section 4.3.4).

Weak rectangle of influence drawings were first studied by Biedl, Bretscher, and Meijer [BBM99]. They showed that a planar graph admits a weak closed rectangle of influence drawing if and only if it admits a planar embedding where the outerface is not a 3-cycle and such that there is no separating 3-cycle; they call a separating 3-cycle a *filled triangle* and call the family of graphs with no filled triangles $NF3$*-graphs*. In the same paper, Biedl, Bretscher, and Meijer also showed that every $NF3$-graph admits an open weak rectangle of influence drawing but left as open the question of characterizing the open weak rectangle of influence drawable graphs. There are several subsequent papers that present partial answers to this question.

Miura, Matsuno, and Nishizeki [MMN09] characterize those *triangulated plane graphs* (i.e., maximal planar graph with a given planar embedding) that admit an open weak rectangle of influence drawing; the characterization gives rise to a linear time testing algorithm. In addition, the paper gives a sufficient condition for the weak open rectangle of influence drawability of *inner triangulated plane graphs* (i.e., planar graphs with a given planar embedding and all triangular faces, except the external face that has more than three vertices). This sufficient condition is expressed in terms of labeling of angles of a suitable subgraph, called *frame graph*. The frame graph of an inner triangulated plane graph G is obtained by removing all vertices and edges in the proper inside of every maximal filled triangle of G. Testing the sufficient condition, and eventually constructing an open rectangle of influence drawing of G, can be executed in $O(n^{1.5} \log n)$ time. The computed drawing has area $(n-1) \times (n-1)$ and it has the property that every edge of the frame graph is *oblique*, i.e., it is neither vertical nor horizontal. Alamdari and Biedl [AB12] further elaborate on the ideas by Miura, Matsuno, and Nishizeki and characterize the inner triangulated plane graphs that admit a weak open rectangle of influence drawing such that no two vertices of the frame graph have the same x-coordinate or the same y-coordinate. The characterization by Alamdari and Biedl yields an $O(n^{1.5} \log n)$-time testing and drawing algorithm. A recent paper by Alamdari and Biedl [AB] generalizes the characterization for non-aligned frames to all planar graphs with a fixed planar embedding. The paper also shows that if the planar embedding is not fixed, then deciding if a given planar graph has an open weak rectangle of influence drawing is NP-complete. NP-completeness holds even for open weak rectangle of influence drawings with non-aligned frames.

A significant research effort has also been devoted to the area required by weak open and closed rectangle of influence drawings. The construction by Biedl, Bretscher, and Meijer [BBM99] gives rise to weak closed and open rectangle of influence drawings with n vertices on an integer grid of size $(n-1) \times (n-1)$. Sadavisam and Zhang [SZ11] show that an integer grid of size at most $(n-3) \times (n-3)$ is always sufficient and sometimes necessary to compute a weak closed rectangle of influence drawing of an *irreducible triangulation*, i.e., a maximal $NF3$-graph. In the same paper, they also proved an expected area of $(\frac{22n}{27} + \sqrt{n}) \times (\frac{22n}{27} + \sqrt{n})$ for a weak closed rectangle of influence drawing of a random irreducible triangulation. Miura and Nishizeki [MN05] prove that the convex grid drawing computed by the algorithm of Miura, Nakano, and Nishizeki [MNN00, MNN06] is in fact a weak open rectangle of influence drawing; this result implies that a four connected planar graph with n vertices has a weak open rectangle of influence drawing in area $\lceil \frac{n-1}{2} \rceil \times \lfloor \frac{n-1}{2} \rfloor$. Zhang and Vaidya [ZV09a, ZV09b] further improve this bound as follows: (i) An irreducible triangulation with n vertices taken uniformly at random has a weak open rectangle of influence drawing whose area is asymptotically $\frac{11n}{27} \times \frac{11n}{27}$ with high probability, up to an additive error of $O(\sqrt{n})$; (ii) A quadriangulation with n vertices taken uniformly at random has a weak open rectangle of influence drawing whose area is asymptotically $\frac{13n}{27} \times \frac{13n}{27}$ with high probability, up to an additive error of $O(\sqrt{n})$. Both results are proved as applications of previous techniques by Fusy [Fus06, Fus09].

4.4.3 Approximate Proximity Drawings

As discussed in Section 4.3, proximity drawability imposes severe restrictions on the families of the representable graphs; for example, the tables of Section 4.3.6 show families of β-drawable graphs whose maximum vertex degrees are all bounded by small constant values. In order to overcome these restrictions on the combinatorial structure of the drawable graphs, recent papers study straight-line drawings of graphs that are "good approximations" of proximity drawings.

Di Giacomo et al. [DDLM12] investigate drawings that approximate the global proximity rule; in particular, they study approximate minimum weight drawings of trees in the 2-dimensional space. A $(1+\varepsilon)$-*EMST drawing* is a planar straight-line drawing of a tree such that, for any fixed $\varepsilon > 0$, the distance between any two vertices is at least $\frac{1}{1+\varepsilon}$ the length of the longest edge in the path connecting them. Therefore, $(1+\varepsilon)$-EMST drawings are good approximations of Euclidean minimum spanning trees. Figure 4.16 shows a $(1+\varepsilon)$-EMST drawing of a tree for $\varepsilon = 0.5$. In the figure, the ratio between the distance $d(u, v)$ and the length of the longest edge along the path between u and v is $\frac{d(u,v)}{|e_T(u,v)|} = 0.714$, which is larger than $\frac{1}{1+\varepsilon} = 0.667$. Note that the tree of the figure does not admit a minimum weight drawing (a Euclidean minimum spanning tree cannot have two adjacent vertices both having degree six).

While it is known that all trees with maximum vertex degree five have a Euclidean minimum spanning tree realization [MS92] and it is NP-hard deciding whether trees of maximum vertex degree six admit one [EW96b], in [DDLM12] it is shown that every tree T has a $(1+\varepsilon)$-EMST drawing for any given $\varepsilon > 0$ and that this drawing can be computed in linear time in the real RAM model of computation.

Also, while Angelini et al. [ABC$^+$11] have proved that EMST drawings of trees with vertex degree at most five may require exponential area, Di Giacomo et al. describe polynomial area approximation schemes for $(1+\varepsilon)$-EMST drawings: Any tree with n vertices and maximum vertex degree Δ admits a $(1+\varepsilon)$-EMST drawing whose area is $O(n^{c+f(\varepsilon,\Delta)})$, where c is a positive constant and $f(\varepsilon, \Delta)$ is a polylogarithmic function that tends to infinity as ε tends to zero. As already mentioned in Section 4.3.1, a byproduct of the techniques of [DDLM12] ia that the polynomial area upper bound for minimum weight drawings of complete binary trees by Frati and Kaufmann [FK11] is improved from $O(n^{4.3})$ to $O(n^{3.8})$.

Evans et al. [EGK$^+$12], introduce and study approximations of $(h, 0)$-proximity drawings called $(\varepsilon_1, \varepsilon_2)$-*proximity drawings*. Intuitively, given a definition of proximity region and two real numbers $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$, an $(\varepsilon_1, \varepsilon_2)$-proximity drawing of a graph is a planar straight-line drawing Γ such that: (i) For every pair of adjacent vertices u, v, their proximity region "shrunk" by the multiplicative factor $\frac{1}{1+\varepsilon_1}$ does not contain any vertices of Γ; and

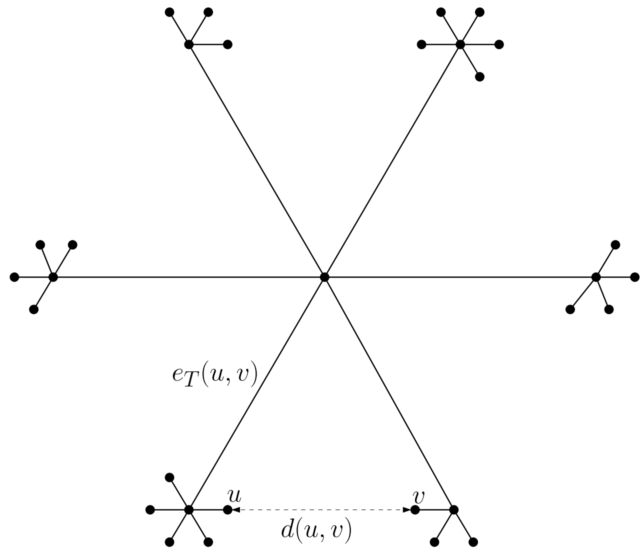


Figure 4.16 A $(1+\varepsilon)$-EMST drawing Γ of a tree with maximum vertex degree 6 for $\varepsilon = 0.5$. For the two highlighted vertices u and v, we have that $\frac{d(u,v)}{|e_T(u,v)|} = 0.714 \geq \frac{1}{1+\varepsilon} = 0.667$.

(ii) For every pair of non-adjacent vertices u, v, their proximity region "blown-up" by the factor $(1+\varepsilon_2)$ contains some vertices of Γ other than u and v. More formally, let D be a disk with center c and radius r, and let ε_1 and ε_2 be two nonnegative real numbers. The ε_1*-shrunk disk of* D is the disk centered at c and having radius $\frac{r}{1+\varepsilon_1}$; the ε_2*-expanded disk of* D is the disk centered at c and having radius $(1+\varepsilon_2)r$. An $(\varepsilon_1, \varepsilon_2)$-proximity drawing is a planar straight-line drawing where the proximity region of two adjacent vertices is defined by using ε_1-shrunk disks, while the region of influence of two non-adjacent vertices uses ε_2-expanded disks.

Figure 4.17 is an example of an $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing for $\varepsilon_1 = 0$ and $\varepsilon_2 = 0.7$. Note that the drawing is not a Gabriel drawing: For example, the dotted disk in the figure is a Gabriel disk (and its emptiness would imply an edge), while the solid one is its 0.7-expanded version. In fact, the tree of Figure 4.17 is not Gabriel drawable (see also Table 4.1).

In [EGK+12], it is proved that one can arbitrarily approximate a proximity drawing of any planar graph for some of the most-studied definitions of proximity. Namely, it is shown that for any positive values of $\varepsilon_1, \varepsilon_2$ an embedded planar graph admits both an $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing and an $(\varepsilon_1, \varepsilon_2)$-Delaunay drawing and an $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing ($1 \leq \beta \leq \infty$) that preserve the given embedding. These results are proved to be, in a sense, tight since it is shown that for each of the above types of proximity rules there are embedded planar graphs that do not have an embedding preserving $(\varepsilon_1, \varepsilon_2)$-proximity drawing with either $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$.

Note that both the strong and the weak proximity drawings described in Sections 4.2.1 and 4.4.2 are special cases of $(\varepsilon_1, \varepsilon_2)$-proximity drawings. Namely, an $(\varepsilon_1, \varepsilon_2)$-proximity drawing is a strong proximity drawing if $\varepsilon_1 = \varepsilon_2 = 0$; also, an $(\varepsilon_1, \varepsilon_2)$-proximity drawing is a weak proximity drawing if $\varepsilon_1 = 0$ and $\varepsilon_2 = \infty$. Therefore, $(0, \varepsilon_2)$-proximity drawings make it possible to study weak and strong proximity drawability in a unified framework: As the value of ε_2 increases, $(0, \varepsilon_2)$-proximity drawings approach weak proximity drawings.

Several questions can be asked within this unifying framework. For example, not all trees have a Gabriel drawing [BLL96], while all trees have a weak Gabriel drawing [DLW06].

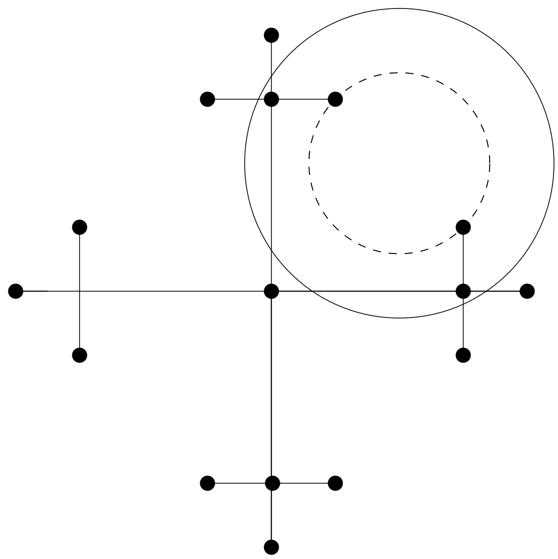


Figure 4.17 A ((0,0.7)-Gabriel drawing of a tree that does not have a Gabriel drawing.

What is the minimum threshold value such that if ε_2 is larger than this threshold all trees are drawable? Evans et al. [EGK$^+$12] answer this question by proving that every tree has a $(0, \varepsilon_2)$-Gabriel drawing for any given value of ε_2 such that $\varepsilon_2 \geq 2$. In the same paper, it is also proved that for each value of ε_2 such that $0 \leq \varepsilon_2 < 2$, there exists a tree T such that T does not have a $(0, \varepsilon_2)$-Gabriel drawing.

All biconnected outerplanar graphs have a Gabriel drawing [LL97], while a connected outerplanar graph where a cut vertex is shared by more than four biconnected components is not Gabriel drawable (see also Section 4.3). For a contrast, it is shown in [EGK$^+$12] that every outerplanar graph without vertices of degree one admits a $(0, \varepsilon_2)$-Gabriel drawings for any arbitrarily chosen positive value of ε_2.

The study of approximate rectangle of influence drawings has also been recently initiated in [DLM], where it is proved that all planar graphs have an open/closed $(\varepsilon_1, \varepsilon_2)$-rectangle of influence drawing for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, while there are planar graphs that do not admit an open/closed $(\varepsilon_1, 0)$-rectangle of influence drawing and planar graphs that do not admit a $(0, \varepsilon_2)$-rectangle of influence drawing. In the same paper, it is shown that all outerplanar graphs have an open/closed $(0, \varepsilon_2)$-rectangle of influence drawing for any $\varepsilon_2 \geq 0$. Concerning area bounds, it is shown that if $\varepsilon_2 > 2$ an open/closed $(0, \varepsilon_2)$-rectangle of influence drawing of an outerplanar can be computed in polynomial area. For values of ε_2 such that $\varepsilon_2 \leq 2$, a drawing algorithm is described that computes $(0, \varepsilon_2)$-rectangle of influence drawings of binary trees in area $O(n^{c+f(\varepsilon_2)})$, where c is a positive constant, $f(\varepsilon_2)$ is a polylogarithmic function that tends to infinity as ε_2 tends to zero, and n is the number of vertices of the input tree.

We conclude the section by recalling a different approach, studied by Hurtado et al. [HLW10], to approximate a proximity drawing. Given a graph G, the idea is to first partition G into subgraphs such that each subgraph is proximity drawable and then compute a drawing Γ of G such that each subdrawing of Γ representing a subgraph of the partition is a proximity drawing. In particular, Hurtado et al. showed different drawing techniques that receive as input a tree T with a partition into subtrees of bounded degree and produce as output a drawing of T such that the subdrawing of each subtree is a minimum spanning tree. In a companion paper, Wood [Woo10] studied how to efficiently partition a tree into subtrees of bounded degree.

4.5 Open Problems

To date, a full understanding of the combinatorial properties of the vast majority of proximity drawable graphs is still an elusive goal and the results presented in the previous sections can be regarded as just the first steps moved into this fascinating wide-open research area. We list below some of the possible research directions that in our opinion are among the most interesting.

Minimum Weight Drawings: Characterizing minimum weight drawable triangulations seems to be a serious challenge; a probably less ambitious goal could be to characterize those minimum weight drawable triangulations whose skeleton is a tree. Another interesting open problem for these types of proximity drawings is determining the computational complexity of deciding whether a tree with vertices of degree at most twelve can be drawn as a Euclidean minimum spanning tree in 3-dimensional space. Also, as described in Section 4.3.1, the algorithm by Monma and Suri [MS92] requires $O(2^{n^2}) \times O(2^{n^2})$ area for a 2-dimensional minimum weight drawing of a tree with n vertices and vertex degree at most five. Angelini et al. [ABC+11] establish an $\Omega(2^n) \times \Omega(2^n)$ lower bound for these trees and conjecture that there is a tree requiring $\Omega(2^{n^2}) \times \Omega(2^{n^2})$. Proving/disproving this conjecture is a fascinating question.

Delaunay and Voronoi Drawings: Characterizing Delaunay drawable graphs is one of the oldest open problems in this area. It would also be interesting to better understand the combinatorial relationship between minimum weight and Delaunay drawable triangulations. Indeed, while Figure 4.9 shows a Delaunay forbidden graph that is minumum weight drawable, it is not known whether there exist Delaunay drawable graphs that are minimum weight forbidden. Another research direction is to study graphs that admit a Delaunay drawing of order h for some $h > 0$; good starting points for this problem are the papers by Abrego et al. [ÁMFM+11] and by Bose et al. [BCH+10], devoted to the combinatorial properties of higher-order proximity graphs. Finally, a complete characterization of (positive or negative) witness Delaunay drawable graphs is another fascinating question.

β-Drawings: The entries of Tables 4.1, 4.2, and 4.3 show gaps in the characterization of strong β-drawable trees and outerplanar graphs. Each of these gaps motivates further research. Also, little is known about the β-drawability properties of general graphs; for example, finding a complete characterization of β drawable k-outerplanar graphs for a given constant k such that $k \geq 2$ is an interesting problem. It would be also interesting to investigate area/volume bounds for strong and weak proximity drawings, also in the unifying framework of $(0, \varepsilon_2)$-proximity drawings. Finally, a natural question is to extend the study of (positive/negative) witness proximity drawability to the whole spectrum of possible β values.

Sphere of Influence Drawings: There are examples of non-planar graphs that admit a sphere of influence drawing. However, the result by Soss [Sos99a] proves that a sphere of influence drawable graph always has a number of edges that is linear in the number of the vertices. It is however not known whether the upper bound of $15n$ by Soss is tight; Toussaint [Tou05] reports on a conjecture of Avis, who claims that such a tight upper bound could be $9n$. What about approximate sphere of influence drawings? Or witness sphere of influence drawings?

Rectangle of Influence Drawings: Except for the classes of graphs described in [LLMW98], very little is known about recognizing which graphs have admit an (open or closed) strong rectangle of influence drawing. Also, as mentioned in the previous section, it would be interesting to characterize which planar graphs have a weak open rectangle of influence drawing. Similar characterizations can also be studied either in the witness proximity or in the approximate proximity models.

Other Proximity Rules: Several well-known proximity rules are still unexplored from a graph drawing point of view. For example, one could study the γ-drawability problem (see Section 4.2.1) or other proximity rules, not mentioned in the previous sections. A very limited list includes α-*complexes* (see, e.g., [Ede95] and also [SLL$^+$08] for preliminary results on α-drawability), *sphere-of-attraction graphs* (see, e.g., [MW00]), *class-cover catch digraphs* (see, e.g., [PMDS03]), and *maximum weight triangulations* (see, e.g., [WCY99, QW04, QW06]).

4.6 Beyond this Chapter

We conclude this chapter by briefly pointing at two research directions in the areas of sensor networks and of robust geometric computing where proximity graphs and drawings have received some attention in the last few years.

Proximity Drawings and Ad-Hoc Networks: Different types of proximity graphs have attracted the interest of network engineers. Indeed, topology control and management, i.e., how to maintain network connectivity while consuming the minimum possible power, has emerged as one of the most important issues in wireless networks.

A wireless sensor network can be modeled as a set of points in the plane where each sensor s can communicate directly with each other sensor that is within its power range; this model gives rise to a proximity graph called a *unit distance graph*, where the proximity region for a sensor s is a circle of radius one centered at s, and there is an edge connecting s to another sensor t if and only if t is within the power range of S. However, the unit distance graph may be too dense for the limited memory of the sensors in the network; also, in order to reduce energy consumption, it is desirable that each sensor communicates directly with only a few of the sensors that are within its range.

An increasing number of topology control algorithms have thus been presented in the literature that are based on proximity graphs that are sparser than the unit distance graph, have small vertex degree, can be computed locally in a distributed manner, and are good spanners (a straight-line drawing Γ of a graph G is a k-*spanner* if for every pair of vertices u and v of G their geometric distance in Γ is at most k times the graph theoretic distance of u and v in G). A limited list of these structures includes k-*localized Delaunay triangulations* (see, e.g., [LCWW03]), *local minimum spanning trees* (see, e.g., [LHS03, CISRS05]), *partial Delaunay triangulations* (see, e.g., [LSW04]), *directed relative neighborhood graphs*, and *directed local minimum spanning trees* [LH04]. The interested reader is also referred to [BM04, BDEK06, BDL$^+$11, CKLS10, CKX11, GLN02, Kan09, KPX10, NS07, Li04] for a limited list of references on geometric spanners and applications of proximity graphs to wireless networks. See also [CBF$^+$06] for a paper that studies the drawability of a graph as a local minimum spanning tree.

We only remark here that all the proximity graphs mentioned above are constructed by pruning those edges of the unit distance graph which do not satisfy a given proximity rule; hence, the resulting proximity drawing guarantees closeness among adjacent vertices while there is no constraint on pairs of non-adjacent vertices. In other words, these structures inherently adopt a weak model of proximity.

Finally, there is general consensus that the knowledge of the combinatorial properties of the communication network is a basic requirement for the design of efficient localized routing algorithms (see, e.g., [BMSU01, KWZ03, LSW05]). Unlike traditional wired and cellular networks, the movement of wireless devices during the communication could change the network topology to some extent: Understanding what types of networks (proximity drawings) can result is therefore a natural question to ask. See, for example, [PS04], where the edge complexity of locally Delaunay triangulations is studied.

Proximity Drawings and Geometric Checkers: The intrinsic structural complexity of the implementation of geometric algorithms makes the problem of formally proving the correctness of the code unfeasible in most of the cases. This has been motivating research on *checkers*. A checker is an algorithm that receives as input a geometric structure and a predicate stating a property that should hold for the structure. The task of the checker is to verify whether the structure satisfies or not the given property. Here, the expectation is that it is often easier to evaluate the quality of the output than the correctness of the software that produces it. Different papers (see, e.g., [DLPT98, MNS$^+$99]) have agreed on the basic features that a "good" checker should have:

Correctness: The checker should be correct beyond any reasonable doubt. Otherwise, one would incur into the problem of checking the checker.

Simplicity: The implementation should be straightforward.

Efficiency: The expectation is to have a checker that is not less efficient than the algorithm that produces the geometric structure.

Robustness: The checker should be able to handle degenerate configurations of the input and should not be affected by errors in the flow of control due to round-off approximations.

Geometric checkers can be quite naturally studied in the context of proximity drawings. Suppose one is given a straight-line drawing Γ of a graph together with some proximity rule $\mathcal{R}$. A *proximity drawing checker* for Γ is an algorithm that either certifies that Γ satisfies the proximity rule $\mathcal{R}$ or reports evidence that Γ does not satisfy $\mathcal{R}$.

One possible approach to solve this problem is to compute the proximity graph on the vertex set of Γ by applying the proximity rule $\mathcal{R}$ and then verify whether the computed drawing coincides with Γ. For example, suppose that Γ is a drawing of a binary tree and one wants to check whether Γ is a minimum weight drawing. One could compute the Euclidean minimum spanning tree of the vertices of Γ in $O(n \log n)$ time [PS90] and verify whether the computed graph coincides with Γ. However, can one perform the check in $o(n \log n)$ time? Also, what if the proximity graph on the vertex set of Γ is not unique? Linear-time checkers for Delaunay and Voronoi drawings can be found in [DLPT98, MNS$^+$99]. Aronov, Dulieu, and Hurtado [ADH] show an $O(n^2 \log m)$-time algorithm that receives as input a straight-line drawing Γ with n vertices and m edges and checks whether Γ is a negative witness Gabriel drawing for some set of witness points. If the answer is affirmative, the algorithm also returns the witness points.

Acknowledgments

This chapter extends and updates an early survey on proximity drawings co-authored by Giuseppe Di Battista, William Lenhart, and me [DLL95]. I thank Boris Aronov, Ferran Hurtado, and Sue H. Whitesides for their insights and constructive comments on earlier versions of this chapter. Work supported in part by MIUR of Italy under project AlgoDEEP prot. 2008TFBWL4.

References

[AB] Soroush Alamdari and Therese Bied. Open rectangle-of-influence drawings of non-triangulated planar graphs. In W. Didimo and M. Patrignani, editors, *Graph Drawing (Proc. 20th International Symposium, GD 2012)*, Lecture Notes Comput. Sci. to appear.

[AB12] Soroush Alamdari and Therese C. Biedl. Planar open rectangle-of-influence drawings with non-aligned frames. In Marc J. van Kreveld and Bettina Speckmann, editors, *Graph Drawing (Proc. 19th International Symposium, GD 2011)*, volume 7034 of *Lecture Notes Comput. Sci.*, pages 14–25. Springer-Verlag, 2012.

[ABC+11] Patrizio Angelini, Till Bruckdorfer, Marco Chiesa, Fabrizio Frati, Michael Kaufmann, and Claudio Squarcella. On the area requirements of Euclidean minimum spanning trees. In Frank Dehne, John Iacono, and Jörg-Rüdiger Sack, editors, *Algorithms and Data Structures (Proc. 12th International Symposium, WADS 2011)*, volume 6844 of *Lecture Notes Comput. Sci.*, pages 25–36. Springer-Verlag, 2011.

[ADH] Boris Aronov, Muriel Dulieu, and Ferran Hurtado. Witness Gabriel graphs. *Computational Geometry.* to appear.

[ADH11a] Boris Aronov, Muriel Dulieu, and Ferran Hurtado. Witness (Delaunay) graphs. *Comput. Geom.*, 44(6-7):329–344, 2011.

[ADH11b] Boris Aronov, Muriel Dulieu, and Ferran Hurtado. Witness rectangle graphs. In Frank Dehne, John Iacono, and Jörg-Rüdiger Sack, editors, *Algorithms and Data Structures (Proc. 12th International Symposium, WADS 2011)*, volume 6844 of *Lecture Notes Comput. Sci.*, pages 73–85. Springer-Verlag, 2011.

[AH82] D. Avis and J. Horton. Remarks on the sphere of influence graphs. *Ann. New York Acad. Sci.*, 440:323–327, 1982.

[AM92] Pankaj K. Agarwal and J. Matoušek. Relative neighborhood graphs in three dimensions. *Comput. Geom. Theory Appl.*, 2(1):1–14, 1992.

[ÁMFM+11] Bernardo M. Ábrego, Ruy Fabila Monroy, Silvia Fernández-Merchant, David Flores-Peñaloza, Ferran Hurtado, Vera Sacristan, and Maria Saumell. On crossing numbers of geometric proximity graphs. *Comput. Geom.*, 44(4):216–233, 2011.

[BBM99] T. Biedl, A. Bretscher, and H. Meijer. Rectangle of influence drawings of graphs without filled 3-cycles. In *Graph Drawing (Proc. GD '99)*, volume 1731 of *Lecture Notes Comput. Sci.*, pages 359–368. Springer-Verlag, 1999.

[BC87] S. Bhatt and S. Cosmadakis. The complexity of minimizing wire lengths in VLSI layouts. *Inform. Process. Lett.*, 25:263–267, 1987.

[BCH+10] Prosenjit Bose, Sébastien Collette, Ferran Hurtado, Matias Korman, Stefan Langerman, Vera Sacristan, and Maria Saumell. Some properties of higher order Delaunay and Gabriel graphs. In *Proceedings of the 22nd Annual Canadian Conference on Computational Geometry, CCCG 2010*, pages 13–16, 2010.

[BDEK06] Prosenjit Bose, Luc Devroye, William S. Evans, and David G. Kirkpatrick. On the spanning ratio of Gabriel graphs and beta-skeletons. *SIAM J. Discrete Math.*, 20(2):412–427, 2006.

[BDL+11] Prosenjit Bose, Luc Devroye, Maarten Löffler, Jack Snoeyink, and Vishal Verma. Almost all Delaunay triangulations have stretch factor greater than $pi/2$. *Comput. Geom.*, 44(2):121–127, 2011.

[BDLL95] P. Bose, G. Di Battista, W. Lenhart, and G. Liotta. Proximity constraints and representable trees. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes Comput. Sci.*, pages 340–351. Springer-Verlag, 1995.

[BE51] P. Bateman and P. Erdös. Geometrical extrema suggested by a lemma of Besicovitch. *American Mathematical Monthly*, 58:306–314, 1951.

[Bes45] A.S. Besicovitch. A general form of the covering principle and relative differentiation of additive functions. *Proceedings of the Cambridge Philosophical Society*, 41:103–110, 1945.

[BLL96] P. Bose, W. Lenhart, and G. Liotta. Characterizing proximity trees. *Algorithmica*, 16:83–110, 1996. (special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia).

[BLS00] E. Boyer, L. Lister, and B. Shader. Sphere of influence graphs using the sup-norm. *Mathematical and Computer Modelling*, 32(10):1071–1082, 2000.

[BM04] Prosenjit Bose and Pat Morin. Online routing in triangulations. *SIAM J. Comput.*, 33(4):937–951, 2004.

[BMSU01] Prosenjit Bose, Pat Morin, Ivan Stojmenovic, and Jorge Urrutia. Routing with guaranteed delivery in ad hoc wireless networks. *Wireless Networks*, 7(6):609–616, 2001.

[Bro79] K. Q. Brown. Voronoi diagrams from convex hulls. *Inform. Process. Lett.*, 9(5):223–228, 1979.

[CBF+06] Pier Francesco Cortese, Giuseppe Di Battista, Fabrizio Frati, Luca Grilli, Katharina Anna Lehmann, Giuseppe Liotta, Maurizio Patrignani, Ioannis G. Tollis, and Francesco Trotta. On the topologies of local minimum spanning trees. In Thomas Erlebach, editor, *Combinatorial and Algorithmic Aspects of Networking, Third Workshop, CAAN 2006*, volume 4235 of *Lecture Notes in Computer Science*, pages 31–44. Springer, 2006.

[CCL09] Jean Cardinal, Sébastien Collette, and Stefan Langerman. Empty region graphs. *Comput. Geom.*, 42(3):183–195, 2009.

[CEG+94] B. Chazelle, H. Edelsbrunner, L. Guibas, J. Hershberger, R. Seidel, and M. Sharir. Selecting heavily covered points. *SIAM J. Comput.*, 23:1138–1151, 1994.

[Cim92] R. J. Cimikowski. Properties of some Euclidian proximity graphs. *Pattern Recogn. Lett.*, 13(6):417–423, June 1992.

[CISRS05] Julien Cartigny, François Ingelrest, David Simplot-Ryl, and Ivan Stojmenovic. Localized LMST and RNG based minimum-energy broadcast protocols in ad hoc networks. *Ad Hoc Networks*, 3(1):1–16, 2005.

[CKLS10] Siu-Wing Cheng, Christian Knauer, Stefan Langerman, and Michiel H. M. Smid. Approximating the average stretch factor of geometric graphs. In Otfried Cheong, Kyung-Yong Chwa, and Kunsoo Park, editors, *Algorithms and Computation - 21st International Symposium, ISAAC 2010*, volume 6506 of *Lecture Notes in Computer Science*, pages 37–48. Springer, 2010.

[CKX11] Shiliang Cui, Iyad A. Kanj, and Ge Xia. On the stretch factor of Delaunay triangulations of points in convex position. *Comput. Geom.*, 44(2):104–109, 2011.

[CPZ04] Miguel Á. Carreira-Perpiñán and Richard S. Zemel. Proximity graphs for clustering and manifold learning. In *Neural Information Processing Systems, NIPS 2004*, 2004.

[CTL92] M. S. Chang, C. Y. Tang, and C. T. Lee. Solving the Euclidean bottleneck matching problem by k-relative neighborhood graphs. *Algorithmica*, 8:177–194, 1992.

[CWL02] G. Calinescu, P. Wan, and X. Li. Distributed construction of planar spanners and routing for ad hoc wireless networks. In *Proc. 21st Annual Joint Conference of the IEEE Computer and Communication Societies (INFOCOM 02)*, 2002.

[CX01] Siu-Wing Cheng and Yin-Feng Xu. On β-skeleton as a subgraph of the minimum weight triangulation. *Theoretical Computer Science*, 262:459–471, 2001.

[DDLM12] Emilio Di Giacomo, Walter Didimo, Giuseppe Liotta, and Henk Meijer. Drawing a tree as a minimum spanning tree approximation. *J. Comput. Syst. Sci.*, 78(2):491–503, 2012.

[Del34] B. Delaunay. Sur la sphère vide. A la memoire de Georges Voronoi. *Izv. Akad. Nauk SSSR, Otdelenie Matematicheskih i Estestvennyh Nauk*, 7:793–800, 1934.

[DETT99] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing.* Prentice Hall, Upper Saddle River, NJ, 1999.

[Dil87] M. B. Dillencourt. A non-Hamiltonian, nondegenerate Delaunay triangulation. *Inform. Process. Lett.*, 25:149–151, 1987.

[Dil89] M. B. Dillencourt. An upper bound on the shortness exponent of inscribable polytopes. *J. Combin. Theory Ser. B*, 46(1):66–83, February 1989.

[Dil90a] M. B. Dillencourt. Realizability of Delaunay triangulations. *Inform. Process. Lett.*, 33(6):283–287, February 1990.

[Dil90b] M. B. Dillencourt. Toughness and Delaunay triangulations. *Discrete Comput. Geom.*, 5:575–601, 1990.

[Dil96] M. B. Dillencourt. Finding Hamiltonian cycles in Delaunay triangulations is NP-complete. *Discrete Applied Mathematics*, 64(3):207–217, 1996.

[DLL95] G. Di Battista, W. Lenhart, and G. Liotta. Proximity drawability: a survey. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes Comput. Sci.*, pages 328–339. Springer-Verlag, 1995.

[DLM] Emilio Di Giacomo, Giuseppe Liotta, and Henk Meijer. The approximate rectangle of influence drawability problem. In W. Didimo and M. Patrignani, editors, *Graph Drawing (Proc. 20th International Symposium, GD 2012)*, Lecture Notes Comput. Sci. to appear.

[DLPT98] Olivier Devillers, Giuseppe Liotta, Franco P. Preparata, and Roberto Tamassia. Checking the convexity of polytopes and the planarity of subdivisions. *Comput. Geom. Theory Appl.*, 11:187–208, 1998.

[DLW06] Giuseppe Di Battista, Giuseppe Liotta, and Sue Whitesides. The strength of weak proximity. *J. Discrete Algorithms*, 4(3):384–400, 2006.

[DS94] M. B. Dillencourt and W. D. Smith. Graph-theoretical conditions for inscribability and Delaunay realizability. In *Proc. 6th Canad. Conf. Comput. Geom.*, pages 287–292, 1994.

[DS95] M. B. Dillencourt and W. D. Smith. A linear-time algorithm for testing the inscribability of trivalent polyhedra. *Internat. J. Comput. Geom. Appl.*, 5:21–36, 1995.

[DV96] G. Di Battista and L. Vismara. Angles of planar triangular graphs. *SIAM J. Discrete Math.*, 9(3):349–359, 1996.

[Ede95] Herbert Edelsbrunner. The union of balls and its dual shape. *Discrete & Computational Geometry*, 13:415–440, 1995.

[EGK+12] William Evans, Emden R. Gansner, Michael Kaufmann, Giuseppe Liotta, Henk Meijer, and Andreas Spillner. Approximate proximity drawings. In Marc J. van Kreveld and Bettina Speckmann, editors, *Graph Drawing (Proc. 19th International Symposium, GD 2011)*, volume 7034 of *Lecture Notes Comput. Sci.*, pages 166–78. Springer-Verlag, 2012.

[Epp92] D. Eppstein. The diameter of nearest neighbor graphs. Tech. Report 92-76, Dept. Inform. Comput. Sci., Univ. California, Irvine, CA, July 1992.

[Epp09] David Eppstein. Isometric diamond subgraphs. In Ioannis G. Tollis and Maurizio Patrignani, editors, *Graph Drawing (Proc. 16th International Symposium, GD 2008)*, volume 5417 of *Lecture Notes in Computer Science*, pages 384–389, 2009.

[EPY97] David Eppstein, Mike Paterson, and F. Frances Yao. On nearest-neighbor graphs. *Discrete & Computational Geometry*, 17(3):263–282, 1997.

[EW96a] P. Eades and S. Whitesides. The logic engine and the realization problem for nearest neighbor graphs. *Theoret. Comput. Sci.*, 169:23–37, 1996.

[EW96b] P. Eades and S. Whitesides. The realization problem for Euclidean minimum spanning trees is NP-hard. *Algorithmica*, 16:60–82, 1996. (special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia).

[FK11] Fabrizio Frati and Michael Kaufmann. Polynomial area bounds for mst embeddings of trees. *Comput. Geom.*, 44(9):529–543, 2011.

[Fus06] Éric Fusy. Counting d-polytopes with d+3 vertices. *Electr. J. Comb.*, 13(1), 2006.

[Fus09] Éric Fusy. Transversal structures on triangulations: A combinatorial study and straight-line drawings. *Discrete Mathematics*, 309(7):1870–1894, 2009.

[GHvK02] Joachim Gudmundsson, Mikael Hammar, and Marc J. van Kreveld. Higher order Delaunay triangulations. *Comput. Geom.*, 23(1):85–98, 2002.

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman, New York, NY, 1979.

[GLN02] Joachim Gudmundsson, Christos Levcopoulos, and Giri Narasimhan. Fast greedy algorithms for constructing sparse geometric spanners. *SIAM J. Comput.*, 31(5):1479–1500, 2002.

[GO04] J. E. Goodman and J. O'Rourke, editors. *Handbook of Discrete and Computational Geometry, 2nd Edition*. CRC Press, 2004.

[GPS94] L. Guibas, J. Pach, and M. Sharir. Sphere of influence graphs in higher dimensions. *Colloquia Mathematica Societatis János Bolyai*, 63:131–137, 1994.

[GS69] K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18:259–278, 1969.

[Har69] F. Harary. *Graph Theory*. Addison-Wesley, Reading, Mass., 1969.

[HJLM93] F. Harary, M.S. Jacobson, M.J. Lipman, and F.R. Morris. On abstract sphere of influence graphs. *Mathematical and Computer Modelling*, 17(11):77–83, 1993.

[HLW10] Ferran Hurtado, Giuseppe Liotta, and David R. Wood. Proximity drawings of high-degree trees. *CoRR*, abs/1008.3193, 2010.

[IR07] Mohammad Tanvir Irfan and Md. Saidur Rahman. Computing beta-drawings of 2-outerplane graphs. In M. Kaykobad and Md. Saidur Rahman, editors, *Workshop on Algorithms and Computation 2007 - Proceedings of First WALCOM*, pages 46–61. Bangladesh Academy of Sciences (BAS), 2007.

[IS85] M. Ichino and J. Sklansky. The relative neighborhood graph for mixed feature variables. *Pattern Recognition*, 18(2):161–167, 1985.

[JLM95] M.S. Jacobson, M.J. Lipman, and F.R. Morris. Trees that are sphere of influence graphs. *Appl. Math. Letters*, 8(6):89–93, 1995.

[JT92] J. W. Jaromczyk and G. T. Toussaint. Relative neighborhood graphs and their relatives. *Proc. IEEE*, 80(9):1502–1517, September 1992.

[Kan83] V. Kantabutra. Traveling salesman cycles are not always subgraphs of Voronoi duals. *Inform. Process. Lett.*, 16:11–12, 1983.

[Kan09] Iyad A. Kanj. On spanners of geometric graphs. In Jianer Chen and S. Barry Cooper, editors, *Theory and Applications of Models of Computation, 6th Annual Conference, TAMC 2009*, volume 5532 of *Lecture Notes in Computer Science*, pages 49–58. Springer, 2009.

[Kei94] M. Keil. Computing a subgraph of the minimum weight triangulation. *Comput. Geom. Theory Appl.*, 4:13–26, 1994.

[Kin06] James A. King. Realization of degree 10 minimum spanning trees in 3-space. In *Proceedings of the 18th Annual Canadian Conference on Computational Geometry, CCCG 2006*, 2006.

[KL10] Sanjiv Kapoor and Xiang-Yang Li. Proximity structures for geometric graphs. *Int. J. Comput. Geometry Appl.*, 20(4):415–429, 2010.

[KPX10] Iyad A. Kanj, Ljubomir Perkovic, and Ge Xia. On spanners and lightweight spanners of geometric graphs. *SIAM J. Comput.*, 39(6):2132–2161, 2010.

[KR85] D. G. Kirkpatrick and J. D. Radke. A framework for computational morphology. In G. T. Toussaint, editor, *Computational Geometry*, pages 217–248. North-Holland, Amsterdam, Netherlands, 1985.

[KW01] M. Kaufmann and D. Wagner, editors. *Drawing Graphs*, volume 2025 of *Lecture Notes in Computer Science*. Springer-Verlag, 2001.

[KW04] Matthew Kitching and Sue Whitesides. The three dimensional logic engine. In János Pach, editor, *Graph Drawing (Proc. 12th International*

Symposium, GD 2004), volume 3383 of *Lecture Notes in Computer Science*, pages 329–339. Springer, 2004.

[KWZ03] Fabian Kuhn, Roger Wattenhofer, and Aaron Zollinger. Worst-case optimal and average-case efficient geometric ad-hoc routing. In *Proceedings of the 4th ACM Interational Symposium on Mobile Ad Hoc Networking and Computing, MobiHoc 2003*, pages 267–278. ACM, 2003.

[Lan69] P. M. Lankford. Regionalization: theory and alternative algorithms. *Geogr. Anal.*, 1:196–212, 1969.

[LCWW03] X.Y. Li, G. Calinescu, P.J. Wan, and Y. Wang. Localized Delaunay triangulation with application in ad hoc wireless networks. *IEEE Transactions on Parallel and Distributed Systems*, 14:1035–1047, 2003.

[LD95] Giuseppe Liotta and Giuseppe Di Battista. Computing proximity drawings of trees in the 3-dimensional space. In *Proc. 4th Workshop Algorithms Data Struct.*, volume 955 of *Lecture Notes Comput. Sci.*, pages 239–250. Springer-Verlag, 1995.

[LH04] Ning Li and Jennifer C. Hou. Topology control in heterogeneous wireless networks: Problems and solutions. In *INFOCOM*, 2004.

[LHS03] Ning Li, Jennifer C. Hou, and Lui Sha. Design and analysis of an mst-based topology control algorithm. In *INFOCOM*, 2003.

[Li04] X.Y. Li. Applications of computational geometry in wireless networks. In X. Cheng, X. Huang, and D.-Z. Du, editors, *Ad Hoc Wireless Networking*, pages 197–264. Kluwer Academic Publisher, 2004.

[LL96] W. Lenhart and G. Liotta. Drawing outerplanar minimum weight triangulations. *Inform. Process. Lett.*, 57(5):253–260, 1996.

[LL97] W. Lenhart and G. Liotta. Proximity drawings of outerplanar graphs. In S. North, editor, *Graph Drawing (Proc. GD '96)*, volume 1190 of *Lecture Notes Comput. Sci.*, pages 286–302. Springer-Verlag, 1997.

[LL02] W. Lenhart and G. Liotta. The drawability problem for minimum weight triangulations. *Theoretical Computer Science*, 270:261–286, 2002.

[LLMW98] G. Liotta, A. Lubiw, H. Meijer, and S.H. Whitesides. The rectangle of influence drawability problem. *Comput. Geom. Theory and Applications*, 10(1):1–22, 1998.

[LM03] G. Liotta and H. Meijer. Voronoi drawings of trees. *Comput. Geom. Theory and Applications*, 24(3):147–178, 2003.

[LS93] A. Lubiw and N. Sleumer. Maximal outerplanar graphs are relative neighborhood graphs. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 198–203, 1993.

[LSW04] Xiang-Yang Li, Ivan Stojmenovic, and Yu Wang. Partial Delaunay triangulation and degree limited localized bluetooth scatternet formation. *IEEE Transactions on Parallel and Distributed Systems*, 15(4):350–361, 2004.

[LSW05] Xiang-Yang Li, Wen-Zhan Song, and Weizhao Wang. A unified energy efficient topology for unicast and broadcast. In *Proc. MobiCom'05*, 2005.

[LTTV97] G. Liotta, R. Tamassia, I. G. Tollis, and P. Vocca. Area requirement of Gabriel drawings. In *Algorithms and Complexity (Proc. CIAC' 97)*, volume 1203 of *Lecture Notes Comput. Sci.*, pages 135–146. Springer-Verlag, 1997.

[Mat87] C. Mathieu. Some problems in computational geometry. *Algorithmica*, 2:131–134, 1987.

[MMN09] Kazuyuki Miura, Tetsuya Matsuno, and Takao Nishizeki. Open rectangle-of-influence drawings of inner triangulated plane graphs. *Discrete & Computational Geometry*, 41(4):643–670, 2009.

[MN05] Kazuyuki Miura and Takao Nishizeki. Rectangle-of-influence drawings of four-connected plane graphs. In Seok-Hee Hong, editor, *Asia-Pacific Symposium on Information Visualisation, APVIS 2005*, volume 45 of *CRPIT*, pages 75–80, 2005.

[MNN00] Kazuyuki Miura, Takao Nishizeki, and Shin-Ichi Nakano. Convex grid drwaings of four-connected plane graphs. In D. T. Lee and Shang-Hua Teng, editors, *Algorithms and Computation, 11th International Conference, ISAAC 2000*, volume 1969 of *Lecture Notes in Computer Science*, pages 254–265. Springer, 2000.

[MNN06] Kazuyuki Miura, Shin-Ichi Nakano, and Takao Nishizeki. Convex grid drawings of four-connected plane graphs. *Int. J. Found. Comput. Sci.*, 17(5):1031–1060, 2006.

[MNS+99] K. Mehlhorn, S. Näher, M. Seel, R. Seidel, T. Schilz, S. Schirra, and C. Uhrig. Checking geometric programs or verification of geometric structures. *Comput. Geom. Theory Appl.*, 12(1–2):85–103, 1999.

[MQ94a] T.S. Michael and T. Quint. Sphere of influence graphs: a survey. *Congressus Numerantium*, 105:153–160, 1994.

[MQ94b] T.S. Michael and T. Quint. Sphere of influence graphs: Edge density and clique size. *Mathematical and Computer Modelling*, 127(7):19–24, 1994.

[MQ99] T.S. Michael and T. Quint. Sphere of influence graphs in general metric spaces. *Mathematical and Computer Modelling*, 29(7):45–53, 1999.

[MQ03] T.S. Michael and T. Quint. Sphere of influence graphs and the l_∞ metric. *Discrete Applied Mathematics*, 127:447–460, 2003.

[MR08] Wolfgang Mulzer and Günter Rote. Minimum-weight triangulation is NP-hard. *J. ACM*, 55(2), 2008.

[MS80] D. W. Matula and R. R. Sokal. Properties of Gabriel graphs relevant to geographic variation research and clustering of points in the plane. *Geogr. Anal.*, 12(3):205–222, 1980.

[MS92] C. Monma and Subhash Suri. Transitions in geometric minimum spanning trees. *Discrete Comput. Geom.*, 8:265–293, 1992.

[MW00] F. R. McMorris and C. Wang. Sphere of attraction graphs. *Congressus Numerantium*, 142:149–160, 2000.

[NS07] Giri Narasimhan and Michiel H. M. Smid. *Geometric spanner networks*. Cambridge University Press, 2007.

[OBS92] Atsuyuki Okabe, Barry Boots, and Kokichi Sugihara. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley & Sons, Chichester, UK, 1992.

[O'R87] J. O'Rourke. Computational geometry column 2. *SIGACT News*, 18(2):10–12, 1987. Also in Computer Graphics 21(1987), 155–157.

[PMDS03] C. E. Priebe, D. J. Marchette, J. DeVinney, and D.A. Socolinsky. Classification using class cover catch digraphs. *Journal of Classification*, 20(1):3–23, 2003.

[PS90] F. P. Preparata and M. I. Shamos. *Computational Geometry: An Introduction*. Springer-Verlag, 3rd edition, October 1990.

[PS04] R. Pinchasi and S. Smorodinsky. On locally Delaunay geometric graphs. In *Proc. 20th ACM Symposium on Computational Geometry (SoCG'04)*, pages 378–382, 2004.

[PV04] Paolo Penna and Paola Vocca. Proximity drawings in polynomial area and volume. *Comput. Geom. Theory and Applications*, 29(2):91–116, 2004.

[PY92] M. S. Paterson and F. F. Yao. On nearest-neighbor graphs. In *Proc. 19th Internat. Colloq. Automata Lang. Program.*, volume 623 of *Lecture Notes Comput. Sci.*, pages 416–426. Springer-Verlag, 1992.

[QW04] Jianbo Qian and Cao An Wang. A linear-time approximation scheme for maximum weight triangulation of convex polygons. *Algorithmica*, 40(3):161–172, 2004.

[QW06] Jianbo Qian and Cao An Wang. Progress on maximum weight triangulation. *Comput. Geom.*, 33(3):99–105, 2006.

[Rad88] J. D. Radke. On the shape of a set of points. In G. T. Toussaint, editor, *Computational Morphology*, pages 105–136. North-Holland, Amsterdam, Netherlands, 1988.

[Rei48] E.R. Reifenberg. A problem on circles. *Mathematical Gazette*, 32:290–292, 1948.

[RS95] G. Robins and J. S. Salowe. Low-degree minimum spanning trees. *Discrete Comput. Geom.*, 14:151–165, 1995.

[SH97] Kokichi Sugihara and Tetsuya Hiroshima. How to draw a Delaunay diagram with a given topology. In *Abstracts 13th European Workshop Comput. Geom.*, pages 13–15. Universität Würzburg, 1997.

[SI92] K. Sugihara and M. Iri. Construction of the Voronoi diagram for 'one million' generators in single-precision arithmetic. *Proc. IEEE*, 80(9):1471–1484, September 1992.

[SIII00] K. Sugihara, M. Iri, H. Inagaki, and T. Imai. Topology-oriented implementation - an approach to robust geometric algorithms. *Algorithmica*, 27(1):5–20, 2000.

[SIR08] Md. Abul Hassan Samee, Mohammad Tanvir Irfan, and Md. Saidur Rahman. Computing *beta* -drawings of 2-outerplane graphs in linear time. In Shin-Ichi Nakano and Md. Saidur Rahman, editors, *Algorithms and Computation, Second International Workshop, WALCOM 2008*, volume 4921 of *Lecture Notes in Computer Science*, pages 81–87. Springer, 2008.

[SLL+08] Svetlana Stolpner, Jonathan Lenchner, Giuseppe Liotta, David Bremner, Christophe Paul, Marc Pouget, and Stephen K. Wismath. A note on alpha-drawable k-trees. In *Proceedings of the 20th Annual Canadian Conference on Computational Geometry*, 2008.

[Sos99a] M. Soss. On the size of the Euclidean sphere of influence graph. In *Proc. 11th Canad. Conf. Comput. Geom.*, 1999.

[Sos99b] M. Soss. The size of the open sphere of influence graph in L_∞ metric spaces. In *Graph Drawing (Proc. GD '98)*, volume 1547 of *Lecture Notes Comput. Sci.*, pages 458–459. Springer-Verlag, 1999.

[SZ11] Sadish Sadasivam and Huaming Zhang. Closed rectangle-of-influence drawings for irreducible triangulations. *Comput. Geom.*, 44(1):9–19, 2011.

[Tou80] G. T. Toussaint. The relative neighbourhood graph of a finite planar set. *Pattern Recogn.*, 12:261–268, 1980.

[Tou88] G. T. Toussaint. A graph-theoretical primal sketch. In G. T. Toussaint, editor, *Computational Morphology*, pages 229–260. North-Holland, Amsterdam, Netherlands, 1988.

[Tou05] G. Toussaint. Geometric proximity graphs for improving nearest neighbor methods in instance-based learning and data mining. *International Journal of Comput. Geom. and Applications*, 15(2):101–150, 2005.

[Urq83] R. B. Urquhart. Some properties of the planar Euclidean relative neighbourhood graph. *Pattern Recogn. Lett.*, 1:317–332, 1983.

[Vel92] R. C. Veltkamp. The γ-neighborhood graph. *Comput. Geom. Theory Appl.*, 1(4):227–246, 1992.

[Vel94] R. C. Veltkamp. *Closed Object Boundaries from Scattered Points*, volume 885 of *Lecture Notes Comput. Sci.* Springer-Verlag, 1994.

[Vel95] R. C. Veltkamp. Boundaries through scattered points of unknown density. *Graphics Models and Image Processing*, 57(6):441–452, November 1995.

[WCY99] Cao An Wang, Francis Y. L. Chin, and Bo-Ting Yang. Maximum weight triangulation and graph drawing. *Inf. Process. Lett.*, 70(1):17–22, 1999.

[WCY00] C. An Wang, F. Y. Chin, and B. Yang. Triangulations without minimum weight drawing. *Information Processing Letters*, 74(5–6):183–189, 2000.

[Woo10] David R. Wood. Partitions and coverings of trees by bounded-degree subtrees. *CoRR*, abs/1008.3190, 2010.

[WY01] C. A. Wang and B. Yang. A lower bound for β-skeleton belonging to minimum weight triangulations. *Comput. Geom. Theory Appl.*, 19:35–46, 2001.

[ZV09a] Huaming Zhang and Milind Vaidya. On open rectangle-of-influence and rectangular dual drawings of plane graphs. *Discrete Mathematics, Algorithms and Applications*, 1(3):319–333, 2009.

[ZV09b] Huaming Zhang and Milind Vaidya. On open rectangle-of-influence drawings of planar graphs. In Ding-Zhu Du, Xiaodong Hu, and Panos Pardalos, editors, *Combinatorial Optimization and Applications*, volume 5573 of *Lecture Notes in Computer Science*, pages 123–134. Springer Berlin / Heidelberg, 2009.