The Approximation Complexity of Win-Lose Games

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Abstract

We further our algorithmic and structural understanding of Nash equilibria. Specifically:

- We distill the hard core of the complexity of Nash equilibria, showing that even correctly computing a logarithmic number of bits of the equilibrium strategies of a two-player win-lose game is as hard as the general problem.
- We prove the following structural result about Nash equilibria: "the set of approximate equilibria of a zero-sum game is convex."[1]

1 Introduction

The notion of *Nash equilibria* [18, 19] has captured the imagination of much of the computer science theory community, both for its many applications in the growing domain of online interactions and for its deep and fundamental mathematical structure. As the complexity and scale of typical internet applications increase, the problem of efficiently analyzing their game-theoretic properties becomes more pointed. A variety of algorithms have been proposed to compute Nash equilibria using ideas from mathematical programming [10, 14, 15, 17, 23, 24]. In the last few years, significant progress has been made on algorithms for the approximation of equilibria [16], algorithms that apply to special forms of games [21, 2, 13]. There has been much recent work on characterizing the complexity of the computation and approximation of Nash equilibria [7, 11, 4, 3, 5, 1, 6, 12, 9, 8, 22].

Intuitively, the complexity of a game grows along a few axes: the number of players involved, the number of choices available to each player, the complexity of the payoffs that specify the game, and the accuracy we desire for the computed equilibrium. In this paper we show, however, that being able to compute Nash equilibria is an *all-or-nothing* problem. That is, if we can even roughly approximate the Nash equilibria of the simplest imaginable games—the two-player games where the *outcome* for either player is either a "win" or a "loss"—then we can compute the Nash equilibria of arbitrary games to as much precision as we desire. In particular, we show that finding a logarithmic number of bits of a Nash equilibrium of a two player win-lose game is exactly as hard as the general version: computing a polynomial number of bits of an equilibrium in a general r-player game, for any fixed $r \geq 2$.

The consideration of $\Theta(\log n)$-bit computation is significant both from a numerical analysis and a practical modeling perspective. Recently the field of algorithms has shifted focus from *exact* to *approximate* algorithms, in large part because for most purposes an approximate solution is almost as meaningful as an exact solution, but also because in many cases the inputs to the problem are only known approximately. These observations may be particularly true in the realm of game theory, where the parameters of a game are designed to approximately model the true preferences of the players, and the Nash equilibria are meant to model players' actions. In practice, neither the inputs nor the outputs of the Nash problem may be significant beyond the first few digits.

Previously, different papers have analyzed the importance of different aspects of the three "axes of hardness" of the Nash problem: the bit-complexity of the inputs, the number of players, and the desired precision of the output. Abbott, Kane, and Valiant showed that win-lose two-player games are as hard as general two-player games [1]. Chen and Deng [4], building on work by Daskalakis, Goldberg, and Papadimitriou [9] show that two-player games are as hard as r-player games. Chen, Deng and Teng show that finding Nash equilibria of two-player games to $\Theta(\log n)$ bits of precision is as hard as the general two-player case [5]. In this work we combine the strongest aspects of these previous reductions, showing that finding an approximate equilibrium of a two-player win-lose game is as hard as the general case. To prove this, we compose the transformations of [1] and [5], and analyze its stability under approximations. As one of the contributions of this paper, we establish a convexity result about approximate equilibria for two-player zero-sum games, in analogy with the classic result that their exact equilibria form a convex

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[1] See Lemma 3.1 for the precise statement.

set.

2 Games and Nash Equilibria

We introduce the concepts from game theory that we will use for the rest of the paper. We begin with the definition of a two-player game.

DEFINITION 2.1. (TWO-PLAYER GAME) *A two-player game is defined by a pair of real-valued "payoff" or "utility" matrices* $(\mathbf{U}, \mathbf{V})$ *of the same size, where the game is conducted as follows: the first and second players simultaneously and independently choose a row and column respectively of the matrices; the first player receives "payoff" corresponding to the entry of* $\mathbf{U}$ *specified by this row and column, and the second player receives "payoff" corresponding to the entry of* $\mathbf{V}$.

When playing such a game, the players may pick a certain row or column ahead of time and play it with probability 1, or they can pick probability distributions on the rows or columns and, when asked to play, flip random coins and draw from these distributions. We denote a pair of strategies by a pair of vectors $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is a probability distribution over the rows, and $\mathbf{y}$ is a probability distribution over the columns. We will denote the set of probability vectors in n dimensions by $\mathbb{P}^n$.

Given that the first player plays strategy $\mathbf{x}$, if the second player plays column i, then if we let V_i denote the ith column of matrix $\mathbf{V}$, the *expected* payoff the second player will receive is exactly $\mathbf{x}^T V_i$. More generally, we refer to the vector $\mathbf{x}^T\mathbf{V}$ as the vector of *incentives* for the second player to play each of the columns. Thus if the second player plays strategy $\mathbf{y}$, then his expected payoff is $\mathbf{x}^T\mathbf{V}\mathbf{y}$. Similarly, we say that vector $\mathbf{U}\mathbf{y}$ represents the first player's incentives, and the expected payoff for the first player when playing his strategy $\mathbf{x}$ is $\mathbf{x}^T\mathbf{U}\mathbf{y}$.

To capture the notion of *reasonable play* in such a game, we have the notion of a *Nash equilibrium*, which says intuitively that a pair $(\mathbf{x}, \mathbf{y})$ is in equilibrium if each player's strategy is optimal with respect to the other player's strategy. (Note that the next definition is syntactically different from the standard one; we write it this way to motivate the notion of approximate equilibria that we use.)

DEFINITION 2.2. (NASH EQUILIBRIUM) *Given a two-player game* $(\mathbf{U}, \mathbf{V})$, *and denoting the* i*th row of* $\mathbf{U}$ *by* U_i, *and the* i*th column of* $\mathbf{V}$ *by* V_i, *a pair of strategies* $(\mathbf{x}, \mathbf{y})$ *is a* Nash equilibrium *if for each pair of indices* i, j, $U_i\mathbf{y} < U_j\mathbf{y} \Rightarrow x_i = 0$ *and* $\mathbf{x}^T V_i < \mathbf{x}^T V_j \Rightarrow y_i = 0$.

The celebrated theorem of Nash [18, 19] shows that every two-player game $(\mathbf{U}, \mathbf{V})$ has Nash equilibria. The problem of finding one of these has been an exciting and challenging problem. To be precise with the computational complexity, naturally, we have to properly define the input size. In the complexity theory over the reals, each payoff entry is a number. So the input size is just twice the product of the number of row and column strategies. But in the discrete complexity setting, we may assume entries are rational. So, in addition to the number of strategies, the input length of a two-player game depends on the representation of its payoff entries. One can express each rational number $c = d/e$ by the binary representations of d and e, and hence its length is $\lceil \log_2 d \rceil + \lceil \log_2 e \rceil$. The *input* size of a bimatrix game is then the sum of the lengths of all its entries.

Although its worst-case complexity is exponential [22], the classic Lemke-Howson algorithm [15] for finding a Nash equilibrium of a bimatrix game can be used to prove that the entries of equilibria of any rational bimatrix game are rational. Moreover, the length of the rational expression of an equilibrium is polynomial in the input length.

We will refer to the problem of finding the rational representation of an equilibrium of a rational bimatrix game as `RATIONAL BIMATRIX`. A win-lose game on the other hand is specified by a pair of $\{0, 1\}$-matrices. We denote the problem of finding a rational representation of a Nash equilibrium of a win-lose game by `WIN-LOSE BIMATRIX`.

In this paper we are concerned with *approximate* Nash equilibria. There are several ways to define an approximate Nash equilibrium. Perhaps, the most commonly used definition of approximate Nash equilibria is the following one:

DEFINITION 2.3. (ϵ-APPROXIMATE NASH EQUILIBRIUM) *Given a two-player game* $(\mathbf{U}, \mathbf{V})$, *we define a pair of strategies* $(\mathbf{x}, \mathbf{y})$ *to be an* ϵ-approximate Nash equilibrium *if* $\forall\ \mathbf{u} \in \mathbb{P}^n, \mathbf{v} \in \mathbb{P}^n$, *the pair satisfies* $\mathbf{x}^T\mathbf{U}\mathbf{y} \geq \mathbf{u}^T\mathbf{U}\mathbf{y} - \epsilon$ *and* $\mathbf{x}^T\mathbf{V}\mathbf{y} \geq \mathbf{x}^T\mathbf{V}\mathbf{v} - \epsilon$.

For the purpose of convenience, we work with the following definition, which is shown equivalent up to polynomial factors to the above definition in [5].

DEFINITION 2.4. (ϵ-WELL-SUPPORTED NASH) *Given a two-player game* $(\mathbf{U}, \mathbf{V})$, *we define a pair of strategies* $(\mathbf{x}, \mathbf{y})$ *to be an* ϵ-well-supported Nash equilibrium *if for any pair of indices* i, j *we have* $U_i\mathbf{y} < U_j\mathbf{y} - \epsilon \Rightarrow x_i = 0$ *and* $\mathbf{x}V_i < \mathbf{x}V_j - \epsilon \Rightarrow y_i = 0$.

3 A Convexity Lemma

The reduction of [1] involves constructing a 0-1 "generator" game G whose unique Nash equilibrium consists of vectors containing powers of 2; they then use the game G to express arbitrary rational payoffs in binary. In this section and the next we extend their results to show that an approximate version of this result holds even for ϵ-well-supported Nash equilibria.

The result of this section is a convexity lemma we develop for ϵ-well-supported Nash equilibria. Recall from standard game theory that in a so-called *zero-sum* game, where the sum $\mathbf{U} + \mathbf{V} = 0$ throughout the matrix, the Nash equilibria are solutions to a linear program, and hence form a convex set. We show an analog of this result in the approximate equilibrium setting. We use this in the following section to help characterize the approximate equilibria of G. There are very few known results about the general structure of approximate equilibria and we hope that this lemma will prove useful as this field grows.

Lemma 3.1. (Approximate Convexity) *Given a zero-sum bimatrix game $(\mathbf{U}, \mathbf{V})$, a convex combination of a pair of ϵ-well-supported Nash equilibria of $(\mathbf{U}, \mathbf{V})$ will be a $\frac{4}{\delta}\epsilon$-well-supported Nash equilibrium of $(\mathbf{U}, \mathbf{V})$, where δ is the minimum non-zero weight found as an element of one of the equilibria where the corresponding element of the other equilibria has weight 0.*

Proof. Suppose $(\mathbf{U}, \mathbf{V})$ has two ϵ-well supported Nash equilibria, $(\mathbf{r}_1, \mathbf{c}_1)$ and $(\mathbf{r}_2, \mathbf{c}_2)$. Without loss of generality, we can write $\mathbf{r}_1$ and $\mathbf{r}_2$ in block form as $\mathbf{r}_1 = [\mathbf{x}, \mathbf{y}_1, \mathbf{0}]$ and $\mathbf{r}_2 = [\mathbf{0}, \mathbf{y}_2, \mathbf{z}]$, where the first block consists of those rows only played in the first equilibrium, the second block consists of those rows played in both equilibriums, and the third block consists of those rows only played in the second equilibrium. Similarly, we can express $\mathbf{c}_1$ and $\mathbf{c}_2$ in block form as $\mathbf{c}_1 = [\mathbf{p}, \mathbf{q}_1, \mathbf{0}]$ and $\mathbf{c}_2 = [\mathbf{0}, \mathbf{q}_2, \mathbf{r}]$. Note that without loss of generality we may ignore those rows and columns never played in either equilibrium.

With this block decomposition in mind, we express the row-player's payoff matrix $\mathbf{U}$ as

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} & \mathbf{I} \end{pmatrix},$$

where the payoff matrix for the column player is just $\mathbf{V} = -\mathbf{U}$ since $(\mathbf{U}, \mathbf{V})$ is a zero-sum game.

Let I_1 be the maximum incentive for the row player in the equilibrium $(\mathbf{r}_1, \mathbf{c}_1)$, namely $\max \mathbf{U}\mathbf{c}_1$, and let $I_2 = \max \mathbf{U}\mathbf{c}_2$ be the maximum incentive for the row player in the equilibrium $(\mathbf{r}_2, \mathbf{c}_2)$. From the definition of an ϵ-well supported Nash equilibrium, we have the following inequalities:

$$I_1 - \epsilon \leq \mathbf{A}\mathbf{p} + \mathbf{B}\mathbf{q}_1 \leq I_1 \tag{3.1}$$
$$I_1 - \epsilon \leq \mathbf{D}\mathbf{p} + \mathbf{E}\mathbf{q}_1 \leq I_1 \tag{3.2}$$
$$\mathbf{G}\mathbf{p} + \mathbf{H}\mathbf{q}_1 \leq I_1 \tag{3.3}$$

$$I_2 - \epsilon \leq \mathbf{E}\mathbf{q}_2 + \mathbf{F}\mathbf{r} \leq I_2 \tag{3.4}$$
$$I_2 - \epsilon \leq \mathbf{H}\mathbf{q}_2 + \mathbf{I}\mathbf{r} \leq I_2 \tag{3.5}$$
$$\mathbf{B}\mathbf{q}_2 + \mathbf{C}\mathbf{r} \leq I_2 \tag{3.6}$$

Now consider the expression $\mathbf{r}_1\mathbf{U}\mathbf{c}_1 \equiv -\mathbf{r}_1\mathbf{V}\mathbf{c}_1$. We interpret this as follows: up to sign change, the *average* incentive for the row player to play a row equals the average incentive for the column player to play a column, where the averages are taken over the distributions induced by $\mathbf{r}_1$ and $\mathbf{c}_1$ respectively.

Since each incentive is within ϵ of the maximum incentive for that player, the average incentives must also be within ϵ of these respective maxima, and we conclude that

$$-\max \mathbf{r}_1\mathbf{V} \leq I_1 \equiv \max \mathbf{U}\mathbf{c}_1 \leq 2\epsilon - \max \mathbf{r}_1\mathbf{V}, \text{ and}$$
$$-\max \mathbf{r}_2\mathbf{V} \leq I_2 \equiv \max \mathbf{U}\mathbf{c}_2 \leq 2\epsilon - \max \mathbf{r}_2\mathbf{V}.$$

Using this, we now write out the inequalities for the column-player's payoffs. Noting that $\mathbf{V} = -\mathbf{U}$, the direction of the inequalities is different from above:

$$I_1 - 2\epsilon \leq \mathbf{x}\mathbf{A} + \mathbf{y}_1\mathbf{D} \leq I_1 + \epsilon \tag{3.7}$$
$$I_1 - 2\epsilon \leq \mathbf{x}\mathbf{B} + \mathbf{y}_1\mathbf{E} \leq I_1 + \epsilon \tag{3.8}$$
$$I_1 - 2\epsilon \leq \mathbf{x}\mathbf{C} + \mathbf{y}_1\mathbf{F} \tag{3.9}$$
$$I_2 - 2\epsilon \leq \mathbf{y}_2\mathbf{E} + \mathbf{z}\mathbf{H} \leq I_2 + \epsilon \tag{3.10}$$
$$I_2 - 2\epsilon \leq \mathbf{y}_2\mathbf{F} + \mathbf{z}\mathbf{I} \leq I_2 + \epsilon \tag{3.11}$$
$$I_2 - 2\epsilon \leq \mathbf{y}_2\mathbf{D} + \mathbf{z}\mathbf{G} \tag{3.12}$$

If we multiply equation (3.2) on the left by $\mathbf{y}_2$, equation (3.3) by $\mathbf{z}$, and add the results, we have that

$$\mathbf{y}_2\mathbf{D}\mathbf{p} + \mathbf{y}_2\mathbf{E}\mathbf{q}_1 + \mathbf{z}\mathbf{G}\mathbf{p} + \mathbf{z}\mathbf{H}\mathbf{q}_1 \leq I_1(\bar{\mathbf{y}}_2 + \bar{\mathbf{z}}),$$

where we use the notation $\bar{\mathbf{v}}$ to denote the sum of the elements of a vector $\mathbf{v}$. If we multiply equation (3.10) on the right by $\mathbf{q}_1$, equation (3.12) by $\mathbf{p}$, and add the results, we have that

$$(I_2 - 2\epsilon)(\bar{\mathbf{p}} + \bar{\mathbf{q}}_1) \leq \mathbf{y}_2\mathbf{D}\mathbf{p} + \mathbf{y}_2\mathbf{E}\mathbf{q}_1 + \mathbf{z}\mathbf{G}\mathbf{p} + \mathbf{z}\mathbf{H}\mathbf{q}_1. \tag{3.13}$$

Comparing these two inequalities reveals that $(I_2 - 2\epsilon)(\bar{\mathbf{p}} + \bar{\mathbf{q}}_1) \leq I_1(\bar{\mathbf{y}}_2 + \bar{\mathbf{z}})$. We note that since $\mathbf{r}_2$ and $\mathbf{c}_1$ are probability vectors, $\bar{\mathbf{p}} + \bar{\mathbf{q}}_1 = \bar{\mathbf{y}}_2 + \bar{\mathbf{z}} = 1$, and we may thus conclude that $I_2 \leq I_1 + 2\epsilon$. Since there is nothing asymmetric about the roles of the row and column players, we have by symmetry that $I_1 \le I_2 + 2\epsilon$. Combining this and (3.13) yields

$$I_1 - 4\epsilon \le \mathbf{y}_2\mathbf{Dp} + \mathbf{y}_2\mathbf{Eq}_1 + \mathbf{zGp} + \mathbf{zHq}_1.$$

From equation (3.2), we have $I_1\bar{\mathbf{y}}_2 \ge \mathbf{y}_2\mathbf{Dp} + \mathbf{y}_2\mathbf{Eq}_1$. Subtracting these two inequalities, and noting that $1 - \bar{\mathbf{y}}_2 = \bar{\mathbf{z}}$ we have that $I_1\bar{\mathbf{z}} - 4\epsilon \le \mathbf{z}(\mathbf{Gp} + \mathbf{Hq}_1)$.

Since each entry of $\mathbf{Gp} + \mathbf{Hq}_1$ is at most I_1 from equation 3, and each entry of $\mathbf{z}$ is at least δ by definition of δ, we conclude that $I_1 - (4\epsilon)/\delta \le \mathbf{Gp} + \mathbf{Hq}_1$, namely that the incentive for the row player to play in rows in the third block when the column player plays strategy $\mathbf{c}_1$ is within $\frac{4}{\delta}\epsilon$ of optimum. Since this is certainly also the case when the column player plays strategy $\mathbf{c}_2$, we have that for any convex combination of strategies $\mathbf{c}_1$ and $\mathbf{c}_2$ the rows of the third row-block have incentive within $\frac{4}{\delta}\epsilon$ of optimal. We note that the same holds true for rows in the second row-block since each incentive is within ϵ of optimal in both the $\mathbf{c}_1$ and $\mathbf{c}_2$ cases by definition.

We now invoke symmetry to claim that the incentives for all the rows and columns of the game are within $\frac{4}{\delta}\epsilon$ of optimal for any convex combination of the two original equilibria, which implies that any convex combination will be a $\frac{4}{\delta}\epsilon$-well supported Nash equilibrium, as desired. ∎

4 Approximate Equilibria in the Generator Game G

Recall from [1] the generator game G, that enabled them to translate general payoffs into binary.

Definition 4.1. (Generator Game) *Define matrices* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{S}^j$ *as*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\mathbf{S}^j = \begin{pmatrix} \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{B} \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{B} & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ \mathbf{A} & \mathbf{B} & \cdots & 0 & 0 \\ \mathbf{B} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The $k \times k$ *matrix* $\mathbf{S}^j$ *is a* $j \times j$ *block matrix, where* $k = 3j$. *Define the game* $G_j = (\mathbf{S}^j, 1 - \mathbf{S}^j)$.

The following claim is shown in [1]:

Claim 4.1. *The game* $G_j = (\mathbf{S}^j, 1 - \mathbf{S}^j)$ *has a unique (exact) Nash equilibrium* $(\mathbf{r}, \mathbf{c})$ *of*

$$\mathbf{r} = \mathbf{c} = \frac{\left(2^{j-1}, 2^{j-1}, 2^{j-1}, ..., 4, 4, 4, 2, 2, 2, 1, 1, 1\right)^T}{3(2^j - 1)}.$$

As a first step to showing an approximate version of this, we characterize the *full-support* approximate equilibria of G_j, that is, those equilibria for which every row and column is played with strictly positive probability.

Claim 4.2. *Every* full-support *ϵ-well-supported Nash equilibrium of the game* $(\mathbf{S}^j, 1 - \mathbf{S}^j)$ *differs from* $(\mathbf{r}, \mathbf{c})$ *defined above by at most* $18j^2\epsilon$ *in any coordinate.*

Proof. Consider such an equilibrium $(\mathbf{r}', \mathbf{c}')$. We note that each of the incentives $\mathbf{S}^j\mathbf{c}'$ must be between q and $q - \epsilon$ where q here is the highest incentive. Let $\mathbf{c}''$ be the solution to the equation $\mathbf{S}^j\mathbf{c}'' = q$. From Claim 4 in [1] we have that $\mathbf{c}''$ must be proportional to our target of $\mathbf{c}$.

We note that $\mathbf{S}^j$ is invertible and further has the property that all the entries of its inverse have absolute value less than 1. The latter can be verified by writing down the closed form of $(\mathbf{S}^j)^{-1}$, which can be quite easily determined from the block-triangular structure of $\mathbf{S}^j$. Thus each coordinate of $\mathbf{c}'$ differs from the corresponding coordinate of $\mathbf{c}''$ by less than $3j\epsilon$, where $3j$ is the number of rows or columns of $\mathbf{S}^j$.

By definition, both $\mathbf{c}$ and $\mathbf{c}'$ have sum 1. Thus $\mathbf{c}''$ has sum within $(3j)^2\epsilon$ of 1, since each of the $3j$ entries of $\mathbf{c}''$ is within $3j\epsilon$ of the corresponding entry of $\mathbf{c}'$. Also, since each of the entries of c is positive, and $\mathbf{c}''$ is proportional to $\mathbf{c}$, the total (L^1) distance between $\mathbf{c}$ and $\mathbf{c}''$ equals the difference between their sums. Thus from the triangle inequality the L^1 distance between $\mathbf{c}$ and $\mathbf{c}'$ is at most $18j^2\epsilon$, as desired.

The same argument applies to the row-player's strategy, yielding the desired result. ∎

We now show that for small enough ϵ we can remove the full-support condition from the above claim without changing the result.

Claim 4.3. *Every ϵ-well supported Nash equilibrium of the game* $(\mathbf{S}^j, 1 - \mathbf{S}^j)$ *differs from* $(\mathbf{r}, \mathbf{c})$ *defined above by at most* $18j^2\epsilon$ *in any coordinate provided* $\epsilon < \frac{1}{648j^2 4^j}$.

Proof. Suppose for the sake of contradiction that we had an equilibrium $(\mathbf{r}', \mathbf{c}')$ which violated this condition. By the previous claim, $(\mathbf{r}', \mathbf{c}')$ cannot have full support: it must play certain strategies with probability 0. We now apply Lemma 3.1 to the pair of equilibria $(\mathbf{r}', \mathbf{c}')$ and $(\mathbf{r}, \mathbf{c})$, with the latter as defined in Claim 4.1.

These are both ϵ-well-supported, so the ϵ from Lemma 3.1 is just ϵ. To compute δ, we note that since $(\mathbf{r}, \mathbf{c})$ is a *full-support* equilibrium, from the definition of δ we have that δ must be at least the minimum value of $\mathbf{r}$ and $\mathbf{c}$, namely $\delta \ge \frac{1}{3(2^j - 1)}$.

We now conclude from Lemma 3.1 that any convex combination of $(\mathbf{r}', \mathbf{c}')$ and $(\mathbf{r}, \mathbf{c})$ is a $12(2^j - 1)\epsilon$-well-supported Nash equilibrium. We note further that any strictly positive convex combination of these will have full support, and is thus subject to Claim 4.2. Explicitly, any strictly positive convex combination of these two equilibria must be within $18j^2 \cdot 12(2^j - 1)\epsilon$ of $(\mathbf{r}, \mathbf{c})$ in any coordinate. Since each entry of $\mathbf{r}$ and $\mathbf{c}$ is at least $\frac{1}{3(2^j-1)}$ and some entry of $(\mathbf{r}', \mathbf{c}')$ is 0, this implies that $\epsilon \geq \frac{1}{648j^2(2^j-1)^2}$, which contradicts our bound on ϵ. Thus no other equilibria exist. ∎

5 Translating to 0-1: Stability Analysis of a Nash Homomorphism

In this section, we introduce a notion of "well-scaled" two-player games and perform a stability analysis of the Nash homomorphism developed in [1] on this family of games. As the family of hard two-player games constructed in [5] can be transformed into well-scaled games, we will use this stability analysis in the next section to prove our main result.

A matrix $\mathbf{U}$ is *K-well-scaled* for a positive integer K if each entry of $\mathbf{U}$ can be expressed as r/K for some integer r between $K/2$ and K. A two-player game $(\mathbf{U}, \mathbf{V})$ is *K-well-scaled* if both $\mathbf{U}$ and $\mathbf{V}$ are K-well-scaled.

The following is the main result of this section.

Theorem 5.1. *There exists a pair of polynomial-time computable functions f, g such that given an $n \times n$ game $H = (\mathbf{A}, \mathbf{B})$, and integers $K = 3(2^k - 1) \leq n$ such that H is K-well-scaled, $f(H)$ is a 0-1 game $H' = (\mathbf{A}', \mathbf{B}')$ of dimensions $\Theta(nk^2)$, and for any ϵ/n^{25}-well-supported Nash equilibrium $(\mathbf{x}', \mathbf{y}')$ of game H', where $\epsilon \leq 1$, $(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}', \mathbf{y}')$ is an ϵ-well-supported Nash equilibrium of H.*

We apply the Nash homomorphism of [1] to construct the map f.

Construction 5.1. (Translation) *Let $H = (\mathbf{A}, \mathbf{B})$ be a bimatrix game, where $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices. $\mathbf{A}$ is K-well-scaled, where $K = 3(2^k - 1) \leq n$, $k \in \mathbb{Z}^+$. Each row of $\mathbf{B}$ has an entry at least $\frac{1}{2}$ and $0 \leq \mathbf{B} \leq 1$. Let $H' = (\mathbf{A}', \mathbf{B}')$ be a pair of $2n \times 2n$ block matrices, defined in Figure 1.*

In detail, the blocks in $\mathbf{A}'$ are: block $(i, 2i-1)$, for $1 \leq i \leq n$, is a $3k \times 1$ vector of all 1s; block $(i, 2i)$, for $1 \leq i \leq n$, is the $3k \times 3k$ matrix $\mathbf{S}^k$; block $(n+i, 2j)$, for $1 \leq i, j \leq n$, is a $1 \times 3k$ 0-1 vector $\mathbf{R}_{i,j}$ such that $\mathbf{R}_{i,j}\mathbf{c} = \mathbf{A}_{i,j}$, and the remaining blocks are matrices of

$$\mathbf{A}' = \begin{pmatrix} 1 & \mathbf{S}^k & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \mathbf{S}^k & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & \mathbf{S}^k \\ 0 & \mathbf{R}_{1,1} & 0 & \mathbf{R}_{1,2} & \dots & 0 & \mathbf{R}_{1,n} \\ 0 & \mathbf{R}_{2,1} & 0 & \mathbf{R}_{2,2} & \dots & 0 & \mathbf{R}_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathbf{R}_{n,1} & 0 & \mathbf{R}_{n,2} & \dots & 0 & \mathbf{R}_{n,n} \end{pmatrix}$$

$$\mathbf{B}' = \begin{pmatrix} 0 & 1-\mathbf{S}^k & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1-\mathbf{S}^k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1-\mathbf{S}^k \\ B_{1,1} & 0 & B_{1,2} & 0 & \dots & B_{1,n} & 0 \\ B_{2,1} & 0 & B_{2,2} & 0 & \dots & B_{2,n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n,1} & 0 & B_{n,2} & 0 & \dots & B_{n,n} & 0 \end{pmatrix}.$$

Figure 1: Matrices A' and B'

all 0's of proper size, where recall the vectors

$$\mathbf{r} = \mathbf{c} = \frac{\left(2^{k-1}, 2^{k-1}, 2^{k-1}, ..., 4, 4, 4, 2, 2, 2, 1, 1, 1\right)^T}{3(2^k - 1)}$$

are the exact *Nash equilibrium of the generator game $G_k = (\mathbf{S}^k, 1 - \mathbf{S}^k)$. Note that for any entry $\mathbf{A}_{i,j}$ of a K-well-scaled game, there exists a 0-1 vector $\mathbf{R}_{i,j}$ such that $\mathbf{R}_{i,j}\mathbf{c} = \mathbf{A}_{i,j}$.*

The blocks of $\mathbf{B}'$ are: block $(i, 2i)$, for $1 \leq i \leq n$, is the $3k \times 3k$ matrix $1 - \mathbf{S}^k$; block $(n+i, 2j-1)$, for $1 \leq i, j \leq n$, is the scalar $B_{i,j}$; the rest of the blocks are matrices of all 0's of proper size.

Let $(\mathbf{x}, \mathbf{y})$ be an ϵ'-well-supported Nash equilibrium of $H' = (\mathbf{A}', \mathbf{B}')$, where $\epsilon' = \epsilon/n^{12}$ and $\epsilon \leq 1$. Since $\mathbf{x}$ and $\mathbf{y}$ may be thought of as block vectors, we index them by block as follows: for $1 \leq i \leq n$ and $1 \leq j \leq 3k$, let $x^i_j = x_{(i-1)(3k)+j}$ and $y^i_j = y_{(i-1)(3k+1)+(j+1)}$. For $1 \leq i \leq n$, let $\mathbf{x}^i_{(*)}$ ($\mathbf{y}^i_{(*)}$) denote the $3k$-vector whose jth entry is x^i_j (y^i_j). Let $\bar{x}^i$ and $\bar{y}^i$ denote the sum of each block, namely

$$\bar{x}^i = \sum_{1 \leq j \leq 3k} x^i_j \quad \text{and} \quad \bar{y}^i = \sum_{1 \leq j \leq 3k} y^i_j.$$

As above, we use $\mathbf{A}'_i$ to denote the ith *row* of $\mathbf{A}'$ and $\mathbf{B}'_i$ to denote the ith *column* of $\mathbf{B}'$. Recall we assumed that $K = 3(2^k - 1) \leq n$, and thus $k = O(\log n)$. We let

$\mathbf{c}^k$ be the vector defined in Section 4 such that $(\mathbf{c}^k, \mathbf{c}^k)$ is the unique Nash equilibrium of generator G_k.

We first prove a few lemmas to characterize the structure of $(\mathbf{x}, \mathbf{y})$, and then we show how this structure is in fact related to that of an approximate equilibrium of the original game H.

LEMMA 5.1. *If* $x_i > 0$ *then* $\mathbf{A}'_i \mathbf{y}^T \geq \frac{1}{n(3k+1)} - \epsilon'$*; If* $y_i > 0$ *then* $\mathbf{B'}_i^T \mathbf{x} \geq \frac{1}{2}\frac{1}{n(3k+1)} - \epsilon'$.

Proof. We note that from the definition of K-well-scaled, and the construction of $\mathbf{A}'$ and $\mathbf{B}'$, we have that each column of $\mathbf{A}'$ contains a 1, and each row of $\mathbf{B}'$ contains an entry at least $\frac{1}{2}$.

From the definition of an ϵ'-well-supported Nash equilibrium we have that any row i with $x_i > 0$ must have incentive $\mathbf{A}'_i \mathbf{y}^T$ within ϵ' of the maximum row incentive. We note that since the game has $n' = n(3k+1)$ columns, one of these columns j must have $y_j \geq 1/n'$. Since this column contains an entry $A_{i',j} = 1$ by the above observation, the i'th row must have incentive at least $1/n'$, and thus any row with $x_i > 0$ must have incentive within ϵ' of this, as desired. The argument for the columns is almost identical. ∎

LEMMA 5.2. *If* $\bar{y}^i \geq n^6\epsilon'$ *for some* $i : 1 \leq i \leq n$*, then we have* $\bar{x}^i \geq 1/n^3$ *and* $\mathbf{y}^i_{(*)} = \bar{y}^i \mathbf{c}^k \pm n^4\epsilon'$.

Proof. Suppose for the sake of contradiction that $\bar{x}^i < 1/n^3$. Then for each column of the block $\mathbf{y}^i_{(*)}$, since $0 \leq B' \leq 1$, the expected payoff of the second player is less than $1/n^3$. But $1/n^3 < \frac{1}{2}\frac{1}{n(3k+1)} - \epsilon'$, we may invoke Lemma 5.2 to conclude that the columns of this block are never played, the desired contradiction.

To prove the second half, we note that if we let $L = \max(1/\bar{x}^i, 1/\bar{y}^i)$, then if we consider the pair of vectors $(\mathbf{x}^i_{(*)}/\bar{x}^i, \mathbf{y}^i_{(*)}/\bar{y}^i)$ as a pair of strategies in the *sub-game* G_k, we note that they satisfy all the conditions for being a $L\epsilon'$-well-supported Nash equilibrium of the game G_k.

Since $\bar{x}^i \geq 1/n^3$ and $\bar{y}^i \geq n^6\epsilon'$, we have $L\epsilon' \leq \max(n^3\epsilon', 1/n^6) \leq 1/n^6 \ll 1/(648k^2 4^k)$. We thus invoke Lemma 4.3 to conclude that $\mathbf{y}^i_{(*)} = \bar{y}^i \mathbf{c}^k \pm 18k^2 L\bar{y}^i\epsilon'$. Since $L\bar{y}^i = \max(1, \bar{y}^i/\bar{x}^i)$, and $\bar{y}^i \leq 1$ and $\bar{x}^i \geq 1/n^3$, we have $L\bar{y}^i \leq n^3$. For large enough n, we have $18k^2 \leq n$ since $k = O(\log n)$, so we have that $\mathbf{y}^i_{(*)} = \bar{y}^i \mathbf{c}^k \pm n^4\epsilon'$. ∎

We note the following immediate corollary:

COROLLARY 5.1. *If* $\bar{y}^i \geq n^6\epsilon'$ *for some* $1 \leq i \leq n$*, then for all* j*,* $1 \leq j \leq 3k$ *we have*

$$\mathbf{R}_{j,i}(\mathbf{y}^i_{(*)})^T = A_{j,i}\bar{y}^i \pm n^5\epsilon', \quad \textit{and}$$
$$\mathbf{S}^k_j(\mathbf{y}^i_{(*)})^T = \bar{y}^i/3k \pm n^5\epsilon',$$

where $\mathbf{S}^k_j$ *denotes the* j*th row of* $\mathbf{S}^k$.

Proof. From the definition of $\mathbf{R}_{j,i}$ that $\mathbf{R}_{j,i}\mathbf{c}^k = A_{j,i}$. Since from Lemma 5.2 we have that $\mathbf{y}^i_{(*)} = \bar{y}^i\mathbf{c}^k \pm n^4\epsilon'$, we have that $\mathbf{R}_{j,i}(\mathbf{y}^i_{(*)})^T = A_{j,i}\bar{y}^i \pm n^4\epsilon' \sum \mathbf{R}_{j,i}$. Since $\mathbf{R}_{j,i}$ is a vector of length $3k \ll n$, and each of whose elements is at most 1, we have that $\mathbf{R}_{j,i}(\mathbf{y}^i_{(*)})^T = A_{j,i}\bar{y}^i \pm n^5\epsilon'$, as desired.

To prove the second part, we again have that $\mathbf{y}^i_{(*)} = \bar{y}^i\mathbf{c}^k \pm n^4\epsilon'$ from Lemma 5.2, and note that $\mathbf{c}_k$ is a full-support Nash equilibrium of the $3k \times 3k$ game $G^k = (\mathbf{S}^k, 1 - \mathbf{S}^k)$, and so $\mathbf{S}^k_j\mathbf{c}_k = \frac{1}{3k}$. Thus, as above, we conclude $\mathbf{S}^k_j(\mathbf{y}^i_{(*)})^T = \bar{y}^i/3k \pm n^5\epsilon'$ as desired. ∎

LEMMA 5.3. *If* $s_i \stackrel{\text{def}}{=} y_{(i-1)(3k+1)+1} = 0$ *for some* $1 \leq i \leq n$ *then* $\bar{y}^i < n^6\epsilon'$.

Proof. Assume $\bar{y}^i \geq n^6\epsilon'$, then from Lemma 5.2 we have that $\bar{x}^i \geq 1/n^3$ and $\mathbf{y}^i_{(*)} = \bar{y}^i\mathbf{c}^k \pm n^4\epsilon'$.

Consider the incentive of the row player to play in row-block i as opposed to some other row. Since $s_i = 0$, the only contributions to the incentive for rows in the ith block come from $\mathbf{y}^i_{(*)}$, and we have from the above corollary that $\mathbf{S}^k_j(\mathbf{y}^i_{(*)})^T = \bar{y}^i/3k \pm n^5\epsilon'$. However, consider his incentive to play row $3kn + i$, namely $\mathbf{R}_{j,i}(\mathbf{y}^i_{(*)})^T = A_{j,i}\bar{y}^i \pm n^5\epsilon'$. Since $A_{j,i} \geq \frac{1}{2}$, we have $\mathbf{R}_{j,i}(\mathbf{y}^i_{(*)})^T \geq \frac{1}{2}\bar{y}^i - n^5\epsilon'$. Since $\bar{y}^i \geq n^6\epsilon'$ we have that the incentive for the first player to play row i is more than ϵ' less than his incentive to play row $3kn+i$, and thus he will not play rows in the ith block in an ϵ'-well-supported Nash equilibrium, which contradicts our result that $\bar{x}^i \geq 1/n^3$. ∎

We next provide lower bounds for the total weights in certain rows and columns.

LEMMA 5.4. *Let* $r_i = x_{3kn+i}$ *for all* $i : 1 \leq i \leq n$*, then* $\sum_{1\leq i\leq n} r_i \geq 1/n^3$.

Proof. Suppose for the sake of contradiction that $\sum r_i < 1/n^3$.

Consider the payoffs in the column-blocks indexed by $s_i = y_{(i-1)(3k+1)+1}$. Since the only non-zero payoffs in these columns for the column player lie in the rows indexed by r_i, and each of these payoffs is at most 1, the total incentive for any of these columns is at most $\sum r_i < 1/n^3$. From Lemma 5.2 we see that this implies that $s_i = 0$ for all i. Since y is a vector with sum 1, we have $\sum \bar{y}^i = 1$. Thus for some row block j, we have $\bar{y}^j \geq 1/n \gg n^6\epsilon'$. However, from Lemma 5.3 we have that each $\bar{y}^j \leq n^6\epsilon'$. Since $\epsilon' = \epsilon/n^{12}$ and $\epsilon \leq 1$, we have the desired contradiction. ∎

LEMMA 5.5. $\sum_{1\leq i\leq n} \bar{y}^i \geq 1/n^3$.

Proof. Suppose for the sake of contradiction that $\sum \bar{y}^i < 1/n^3$.

Consider the incentive for the row player to play in the rows corresponding to $r_i = x_{3kn+i}$. Since the only non-zero entries occur in the columns $\bar{y}^{(*)}$, and each entry is at most 1, we have that the total incentive to play in any of these rows is at most $\sum \bar{y}^i < 1/n^3$.

As above, we compare this with Lemma 5.2 and see that $r_i = 0$ for all i. However, this contradicts Lemma 5.4, implying that in fact $\sum \bar{y}^i \geq 1/n^3$, as desired. ∎

We are now in a position to construct a scheme to recover ϵ-well-supported Nash equilibria of the original game H from ϵ'-well-supported Nash equilibria of the modified game H'.

CONSTRUCTION 5.2. *Given an ϵ'-well-supported equilibrium $(\mathbf{x}', \mathbf{y}')$ of H', where $\epsilon' = \epsilon/n^{12}$ and $\epsilon \leq 1$, define the variables $\bar{x}^i, \bar{y}^i, r_i, s_i$ as above. We construct an ϵ-well-supported equilibrium $(\mathbf{x}, \mathbf{y})$ of H:*

- *Let $C_1 = 1/\sum r_i$, and let $\mathbf{x} = C_1\mathbf{r}$, where $x_i = r_i$ for all $i : 1 \leq i \leq n$.*
- *Let $\mathbf{q}$ be the length n vector such that $q_i = 0$ if $\bar{y}^i < n^6\epsilon'$ and $q_i = \bar{y}^i$ otherwise.*
- *Let $C_2 = 1/\sum q_i$, and let $\mathbf{y} = C_2\mathbf{q}$.*

Given what the above lemmas tell us about the structure of the equilibrium $(\mathbf{x}', \mathbf{y}')$ we are now in a position to prove directly that $(\mathbf{x}, \mathbf{y})$ as constructed is an ϵ-well-supported Nash equilibrium of the original game $H = (\mathbf{A}, \mathbf{B})$. We show first that no column played has incentive more than ϵ less than optimal, and then show the corresponding statement for rows.

LEMMA 5.6. *For all $1 \leq i, j \leq n$, if $\mathbf{x}^T\mathbf{B}_j < \mathbf{x}^T\mathbf{B}_i - \epsilon$ then $y_j = 0$.*

Proof. In the above construction, the columns i, j of $\mathbf{B}$ correspond to columns $(3k+1)(i-1)+1$ and $(3k+1)(j-1)+1$ of $\mathbf{B}'$. Further, because of the way we construct $\mathbf{B}'$, the entries in the last n rows of these columns are identical with the entries of $\mathbf{B}$. Thus we have

$$\begin{aligned} \epsilon &< \mathbf{x}^T\mathbf{B}_i - \mathbf{x}^T\mathbf{B}_j = C_1(\mathbf{r}^T\mathbf{B}_i - \mathbf{r}^T\mathbf{B}_j) \\ &= C_1(\mathbf{r}^T\mathbf{B}_{(3k+1)(i-1)+1} - \mathbf{r}^T\mathbf{B}_{(3k+1)(j-1)+1}), \end{aligned}$$

where this last expression, when divided by C_1, is the incentive difference for these columns in the game H'. From Lemma 5.4 we have that $C_1 \leq n^3$, so thus the incentive difference of these columns of H' is at least $\epsilon/n^3 > \epsilon'$. Thus $r_j = 0$, which from the construction of y implies $y_j = 0$ as desired. ∎

LEMMA 5.7. *For all $1 \leq i, j \leq n$, if $\mathbf{A}_j\mathbf{y}^T < \mathbf{A}_i\mathbf{y}^T - \epsilon$ then $x_j = 0$.*

Proof. Consider the incentives for the row player in the transformed game H'. Let I_i, I_j be the incentives to play in rows $3kn+i, 3kn+j$ respectively. We lower-bound I_i and upper-bound I_j to prove the desired result. We have from Corollary 5.1 that

$$\begin{aligned} I_i &= \sum_{1\leq m\leq n} \mathbf{R}_{i,m}(\mathbf{y}^m_{(*)})^T \geq \sum_{q_m>0} \mathbf{R}_{i,m}(\mathbf{y}^m_{(*)})^T \\ &\geq \sum_{q_m>0} A_{i,m}\bar{y}^m - n^5\epsilon' \geq \mathbf{A}_i\mathbf{y}/C_2 - n^6\epsilon'. \end{aligned}$$

Similarly, from Corollary 5.1 and the construction of y we have $I_j \leq \mathbf{A}_j\mathbf{y}/C_2 + n^7\epsilon' + n^6\epsilon'$. We note that from Lemma 5.5 and the definition of C_2 we have that $C_2 \leq n^4$. Thus $I_i - I_j \gg \epsilon'$ since $\epsilon' = \epsilon/n^{12}$. Thus $r_j = 0$ in the game H', from which we conclude that $x_j = 0$, as desired. ∎

We combine the above lemmas to yield the following:

LEMMA 5.8. *There exists a pair of polynomial-time computable functions (f, g) such that given a $n \times n$ game $H = (\mathbf{A}, \mathbf{B})$, and integers $K = (2^k - 1) \leq n$ such that $\mathbf{A}$ is K-well-scaled, and each row of $\mathbf{B}$ has an entry at least $\frac{1}{2}$ and $0 \leq \mathbf{B} \leq 1$, $f(H)$ is a game $H' = (\mathbf{A}', \mathbf{B}')$ where* **1)**. *both $\mathbf{A}'$ and $\mathbf{B}'$ are $n(3k+1) \times n(3k+1)$ matrices;* **2)**. *$\mathbf{A}'$ is a 0-1 matrix, and every column has at least one nonzero entry;* **3)**. *$\mathbf{B}'$ has entries either 0, 1 or from $\mathbf{B}$. For every ϵ/n^{12}-well-supported Nash equilibrium $(\mathbf{x}', \mathbf{y}')$ of H', where $\epsilon \leq 1$, $(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}', \mathbf{y}')$ is an ϵ-well-supported Nash equilibrium of H.*

Let f and g be the two functions defined by Constructions 5.1 and 5.2 respectively. Then we have the desired result from Lemmas 5.6 and 5.7. We now show our main result, Theorem 5.1.

Proof. [Theorem 5.1] Given a K-well-scaled game $H = (A, B)$, first apply Construction 5.1 to yield a game $H' = (\mathbf{A}', \mathbf{B}')$ where $\mathbf{A}'$ and $\mathbf{B}'$ satisfy the conditions listed in Lemma 5.8. Let $n'' = n(3k+1)$ and $K'' = 3(2^{k+1} - 1) = 2K + 3$. We define the game $H'' = (\mathbf{A}'', \mathbf{B}'')$ where $\mathbf{B}'' \stackrel{\text{def}}{=} \mathbf{A}'^T$ and $\mathbf{A}'' \stackrel{\text{def}}{=} \frac{(K+2)+K\mathbf{B}'}{K''}$. One can check that $\mathbf{A}''$ is K''-well-scaled, and $K'' = 2K + 3 \leq 2n + 3 < n''$. For any $c > 0$, if $(\mathbf{x}'', \mathbf{y}'')$ is a $c/3$-well-supported Nash equilibrium of H'', then $(\mathbf{y}'', \mathbf{x}'')$ is a c-well-supported Nash equilibrium of H'. Now we may apply Construction 5.1 again, to yield a 0-1 game $H''' = (\mathbf{A}''', \mathbf{B}''')$.

Given an ϵ/n^{25}-well-supported Nash equilibrium $(\mathbf{x}''', \mathbf{y}''')$ of H''', we may find an $\epsilon/(3n^{12})$-well-supported Nash equilibrium $(\mathbf{x}'', \mathbf{y}'')$ of game H'' by Construction 5.2, since $k = O(\log n)$ and

$$\frac{\epsilon}{3n^{12}} \cdot \frac{1}{(n'')^{12}} = \frac{\epsilon}{3n^{12}} \cdot \frac{1}{(n(3k+1))^{12}} > \frac{\epsilon}{n^{25}}.$$

The construction from H' to H'' shows that $(\mathbf{x}', \mathbf{y}') = (\mathbf{y}'', \mathbf{x}'')$ is an ϵ/n^{12}-well-supported Nash equilibrium of H'. We now apply Construction 5.2 a final time to recover an ϵ-well-supported equilibrium of H, as desired. ∎

6 Approximate Win or Lose: All or Nothing

To define the computational complexity of finding and approximating Nash equilibria, we must decide the format of representation of the equilibria. Note that each entry in an equilibrium is a number between 0 and 1. One way to specify such a number $0 \le c \le 1$ is to express it using its binary representation $(c_0 \bullet c_1 \cdots c_L \cdots)$, where $c_i \in \{0, 1\}$ and $c = \lim_{i\to\infty} \sum_{i=0} c_i/2^i$.

As the binary representation of some rational numbers may not be finite, we have to round off the numbers in order to use a finite representation, resulting in an approximation. The first L bits $c_0, ..., c_{L-1}$ give us an *L-bit approximation* $\tilde{c}$ of c. We can similarly "express" a Nash equilibrium $(\mathbf{x}, \mathbf{y})$ by the finite binary representation of its entries.

In this discrete representation of equilibria, we can define the computational problem of bimatrix games as: given a bimatrix game $(\mathbf{U}, \mathbf{V})$ and an integer L, find two $n \times L$ binary matrices $(\mathbf{X}, \mathbf{Y})$ such that for each i, $\mathbf{X}[i, *]$ and $\mathbf{Y}[i, *]$ are the first L binary bits, respectively, of the i^{th}-entry of $\mathbf{x}$ and $\mathbf{y}$ in a Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $(\mathbf{U}, \mathbf{V})$. We refer to this computational problem as L-`BIT BIMATRIX`. We use L-`BIT RATIONAL BIMATRIX` to refer to the computational problem when the input instances are rational games. When the input instances are win-lose games, we call it L-`BIT WIN-LOSE BIMATRIX`.

In this section, we prove the main result of this paper.

Theorem 6.1. (All-or-Nothing in Approximating Nash for Win-Lose Games) *For any constant $c > 0$, the problem of finding a $1/n^c$-approximate Nash equilibrium of* `WIN-LOSE BIMATRIX` *is exactly as hard as* `RATIONAL BIMATRIX`. *Therefore,* $(1 + c)\log n$-`BIT WIN-LOSE BIMATRIX` *is exactly as hard as* `RATIONAL BIMATRIX`.

Before proving this theorem, we briefly review previous results along these lines. We use $P \equiv Q$ to denote that two problems P and Q are polynomial-time reducible to each other.

- Abbott, Kane, and Valiant [1] proved `WIN-LOSE BIMATRIX` ≡ `RATIONAL BIMATRIX`.
- Chen and Deng [4], building on the work of Daskalakis, Goldberg and Papadimitriou [12, 9], proved `RATIONAL BIMATRIX` is **PPAD**-complete[2]. This result implies that general two-player games are as hard as general r-players games, for any fixed integer r.
  - Proposition 6.1 below implies that $\Theta(n)$-`BIT RATIONAL BIMATRIX` ≡ `RATIONAL BIMATRIX`.
- Chen, Deng, and Teng [5] established that, for any constant $c > 0$, computing a $1/n^c$-approximate Nash equilibrium of `INTEGER BIMATRIX` remains **PPAD**-complete [5], where in an instance $(\mathbf{U}, \mathbf{V})$ of `INTEGER BIMATRIX` with n strategies, each entry of $(\mathbf{U}, \mathbf{V})$ is an integer of magnitude poly(n).
  - Proposition 6.1 implies that $(1 + c)\log n$-`BIT INTEGER BIMATRIX` ≡ `RATIONAL BIMATRIX`.

6.1 Proof of Theorem 6.1.

We first observe the following simple fact: suppose $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium of game $(\mathbf{U}, \mathbf{V})$ where all payoff entries are between 0 and 1 and there are n row and n column strategies; let $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ be the vectors generated by the first L-bits of entries in $(\mathbf{x}, \mathbf{y})$; then $1 - n2^{-L+1} \le \|\tilde{\mathbf{x}}\|_1, \|\tilde{\mathbf{y}}\|_1 \le 1$ and

$$\begin{aligned} \mathbf{x}^T\mathbf{U}\mathbf{y} - n2^{-L+1} &\le \tilde{\mathbf{x}}^T\mathbf{U}\tilde{\mathbf{y}} \le \mathbf{x}^T\mathbf{U}\mathbf{y}, \\ \mathbf{x}^T\mathbf{V}\mathbf{y} - n2^{-L} &\le \tilde{\mathbf{x}}^T\mathbf{V}\tilde{\mathbf{y}} \le \mathbf{x}^T\mathbf{V}\mathbf{y}. \end{aligned}$$

Therefore we have immediately from the definition of ϵ-approximate Nash equilibria,

Proposition 6.1. *From a solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ of an instance $(\mathbf{U}, \mathbf{V})$ of* L-`BIT BIMATRIX`, *we can obtain an $(n2^{-L+1})$-approximate Nash equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ of the bimatrix game $(\mathbf{U}, \mathbf{V})$.*

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[2] **PPAD** is a complexity class introduced by Papadimitriou [20]. Informally, a problem is **PPAD**-complete if it is exactly as hard as general discrete fixed-point problems. As the results of our paper do not directly need the definition of **PPAD**, we refer interested readers to the original paper of Papadimitriou. The only fact we will need in this paper is that a problem is **PPAD**-complete if it is exactly as hard as `RATIONAL BIMATRIX`.

We note that because the complexity class **PPAD** might not be closed under Karp reductions – reductions given by polynomial-time computable transformation functions – we carefully state our main result in the form: *problem A is exactly as hard as problem B*. In particular, we have evidence that $(1+c)\log n$-`BIT WIN-LOSE BIMATRIX` is not in **PPAD** even though it is exactly as hard as `RATIONAL BIMATRIX` according to Karp reductions.

We now prove our main result.

Proof. [Theorem 6.1] By Proposition 6.1, the second statement of the theorem follows directly from the first statement of the theorem. We first prove the statement for some constant $c > 0$ and then show how to reduce the constant. In [5], the following problem is proved to be **PPAD**-complete, and hence as hard as `RATIONAL BIMATRIX`: Given a game $H = (\mathbf{U}, \mathbf{V})$, where $\mathbf{U}$ and $\mathbf{V}$ are $n \times n$ matrices with integer entries between 0 and $n^{1/2}$, find an n^{-1}-well-supported Nash equilibrium.

Let $H = (\mathbf{U}, \mathbf{V})$ be such a bimatrix game with n strategies. Let k be the largest integer such that $K = 3(2^k - 1) \leq n$, where for large enough n we have $2n^{1/2} < K = \Theta(n)$. We construct a K-well-scaled game $(\mathbf{U}', \mathbf{V}')$ by setting $\forall\ i, j : 1 \leq i, j \leq n$,

$$\mathbf{U}'_{i,j} = \frac{(K+1) + 2\mathbf{U}_{i,j}}{2K} \quad \text{and} \quad \mathbf{V}'_{i,j} = \frac{(K+1) + 2\mathbf{V}_{i,j}}{2K}.$$

Since $K \leq n$, every n^{-2}-well-supported Nash equilibrium of H' is also an n^{-1}-well-supported Nash equilibrium of H. By Theorem 5.1, we can construct a 0-1 game H'' of dimensions $n'' = \Theta(nk^2)$ in polynomial time such that, from every n^{-27}-well-supported Nash equilibrium of the 0-1 game H'', one can compute an n^{-2}-well-supported equilibrium of H', and hence an n^{-1}-well-supported equilibrium of H in polynomial time. We have thus shown that finding an n^{-27}-well-supported Nash equilibrium in an $n \times n$ win-lose game is **PPAD**-complete. Using the polynomial equivalence between ϵ-well-supported Nash equilibrium and ϵ-approximate Nash equilibrium of [5], we know that there exists a constant $c > 2$ such that, the problem of finding an n^{-c}-approximate Nash equilibrium in a win-lose game of dimensions $n \times n$ is also **PPAD**-complete. Following an idea from [5], we show in Lemma 7.1 of Section 7 that we can extend the **PPAD**-completeness to any constant $c > 0$. ∎

7 PPAD-Completeness for Any Constant $c > 0$

LEMMA 7.1. *If there exists a constant $c > 0$ such that, the problem of finding an n^{-c}-approximate Nash equilibrium in an $n \times n$ win-lose bimatrix game is* **PPAD***-complete, then it remains* **PPAD***-complete for any constant $c' > 0$.*

We prove this with a padding argument, showing how to pad the game from size $n \times n$ to size $n'' \times n''$ with large uniform blocks of zeros and ones without significantly changing the equilibrium structure.

Proof. If $c < 2$, then finding an n^{-2}-approximate Nash equilibrium in 0-1 games is harder, and thus it is also complete in **PPAD**. Therefore, we can always assume that $c \geq 2$.

We need to prove the lemma for constant $0 < c' < c$. Let $H = (\mathbf{A}, \mathbf{B})$ be an $n \times n$ win-lose game, then we transform it into a new game $H' = (\mathbf{A}', \mathbf{B}')$ as follows. Here we use $\mathbf{A}_i$ and $\mathbf{B}_i$ to denote the i^{th} column and i^{th} row of $\mathbf{A}$ and $\mathbf{B}$, respectively. For each $i : 1 \leq i \leq n$, if $\mathbf{A}_i = 0$, then $\mathbf{A}'_i = 1$, otherwise, $\mathbf{A}'_i = \mathbf{A}_i$. For each $i : 1 \leq i \leq n$, if $\mathbf{B}_i = 0$, then $\mathbf{B}'_i = 1$, otherwise, $\mathbf{B}'_i = \mathbf{B}_i$. One can verify that any ϵ-approximate Nash equilibrium of H' is also an ϵ-approximate Nash equilibrium of H. Further, every row of $\mathbf{A}'$ and every column of $\mathbf{B}'$ has at least one entry with value 1.

Next we construct a $n'' \times n''$ game $H'' = (\mathbf{A}'', \mathbf{B}'')$ where $n'' = n^{\frac{2c}{c'}}$ as follows. Here $\mathbf{A}''$ and $\mathbf{B}''$ are both 2×2 block matrices with $\mathbf{A}''_{1,1} = \mathbf{A}'$, $\mathbf{B}''_{1,1} = \mathbf{B}'$, $\mathbf{A}''_{1,2} = \mathbf{B}''_{2,1} = 1$ and $\mathbf{A}''_{2,1} = \mathbf{A}''_{2,2} = \mathbf{B}''_{1,2} = \mathbf{B}''_{2,2} = 0$. By definitio H'' is a 0-1 game. Now let $(\mathbf{x}'', \mathbf{y}'')$ be a $1/n''^{c'} = 1/n^{2c}$-approximate Nash equilibrium of $H'' = (\mathbf{A}'', \mathbf{B}'')$. By the definition of ϵ-approximate Nash equilibria, one can show that $0 \leq \sum_{n<n\leq n''} x''_i, \sum_{n<n\leq n''} y''_i \leq n^{1-2c} \ll 1/2$, since we assumed that $c \geq 2$. Letting $a = \sum_{1\leq i\leq n} x''_i$ and $b = \sum_{1\leq i\leq n} y''_i$, we construct a profile of mixed strategies $(\mathbf{x}', \mathbf{y}')$ of H' as follows: $x'_i = x''_i/a$ and $y'_i = y''_i/b$ for all $i : 1 \leq i \leq n$. Since $a, b > 1/2$, $(\mathbf{x}', \mathbf{y}')$ is a $2/n^{2c}$-approximate Nash equilibrium of H', which is also a $1/n^c$-approximate Nash equilibrium of H.

The reduction above shows that the problem of computing an $n^{-c'}$-approximate Nash equilibrium in a $n \times n$ win-lose game is **PPAD**-complete for any constant $c' > 0$. ∎

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