Problem 1

Compute the Fourier transform of each of the following signals.

(a) 
\[ x(t) = \begin{cases} 
1 + \cos(\pi t) & |t| \leq 1 \\
0 & |t| > 1 
\end{cases} \]
\[ X(w) = \frac{2\sin(w)}{w} + \frac{2w\sin(w)}{\pi^2 - w^2} \]

(b) 
\[ x(t) = \begin{cases} 
1 - t^2 & 0 < t < 1 \\
0 & \text{otherwise} 
\end{cases} \]
\[ X(w) = \frac{1}{jw} + \frac{2e^{-jw}}{-w^2} - \frac{2e^{-jw} - 2}{jw^3} \]

Problem 2

For each pair of signals below, compute the convolution \( y(t) = x(t) \ast h(t) \) by (1) calculating the Fourier transforms \( X(w) \) and \( H(w) \), (2) using the convolution property of Fourier transforms to compute \( Y(w) \), and (3) calculating the inverse transform of \( Y(w) \).
\[(a) \quad x(t) = te^{-2t}u(t) \text{ and } h(t) = e^{-4t}u(t)\]

\[X(w) = \frac{1}{(2 + jw)^2}\]

\[H(w) = \frac{1}{4 + jw}\]

\[Y(w) = X(w)H(w) = \frac{1}{4(jw + 4)} - \frac{1}{4(jw + 2)} + \frac{1}{2(jw + 2)^2}\]

Taking the inverse Fourier transform of each term in \(Y(w)\), we obtain:

\[y(t) = \left(\frac{1}{4}e^{-4t} - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right)u(t)\]

\[(b) \quad x(t) = e^{-t}u(t) \text{ and } h(t) = e^{t}u(-t)\]

\[Y(w) = X(w)H(w) = \left[\frac{1}{1 + jw}\right]\left[\frac{1}{1 - jw}\right] = \frac{1/2}{1 + jw} + \frac{1/2}{1 - jw}\]

Taking the inverse Fourier transform, we obtain:

\[y(t) = \frac{1}{2}e^{-|t|}\]

**Problem 3**

The input \(x(t)\) and output \(y(t)\) of a stable and causal LTI system are related by the differential equation

\[\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)\]

Let \(h(t)\) be the impulse response of this system.

(a) Find \(H(w)\).

\[Y(w) = X(w)H(w) \implies H(w) = Y(w)/X(w)\]
Taking the Fourier transform of the differential equation, we get:

\[(jw)^2 Y(w) + (6jw)Y(w) + 8Y(w) = 2X(w)\]

\[((jw)^2 + 6jw + 8)Y(w) = 2X(w)\]

\[H(w) = \frac{2}{-w^2 + 6jw + 8} = \frac{2}{-(w - 2j)(w + 4j)} = \frac{1}{jw + 2} - \frac{1}{jw + 4}\]

(b) Find \(h(t)\).

\[h(t) = (e^{-2t} - e^{-4t})u(t)\]

Problem 4

(a) Consider two LTI systems with impulse responses \(h(t)\) and \(g(t)\). Suppose these systems are inverses of one another. Let \(H(w)\) be the Fourier transform of \(h(t)\) and \(G(w)\) be the Fourier transform of \(g(t)\). What is the relationship between \(H(w)\) and \(G(w)\)?

\[H(w)G(w) = 1\]

(b) Consider the LTI system with Fourier transform

\[H(w) = \begin{cases} 
1 & 2 < |w| < 3 \\
0 & \text{otherwise}
\end{cases}\]

Use the result from part (a) to decide if this system is invertible.

The system can’t be invertible because we would need for example that \(G(5)H(5) = 1\) but \(H(5) = 0\) makes this impossible.
Problem 5

Recall the “echo problem” from homeworks 2 and 3. Consider the case of a room whose acoustic properties can be characterized by a system $S$ with impulse response

$$h(t) = \sum_{k=0}^{\infty} e^{-kT}\delta(t - kT)$$

(a) Determine the Fourier transform of $h(t)$.

$$H(w) = \sum_{k=0}^{\infty} e^{-kT(1+jw)} = \frac{1}{1 - e^{-T(1+jw)}}$$

(b) We want to process the output of the system to remove the echo it introduces. Let $g(t)$ be the impulse response of the inverse of $S$. Determine what the Fourier transform of $g(t)$ must be using the Fourier transform of $h(t)$ computed in part (a).

$$G(w) = 1/H(w) = 1 - e^{-T(1+jw)}$$

(c) Use the result from part (b) to determine $g(t)$.

$$g(t) = \delta(t) - e^{-T}\delta(t - T)$$

Hint for (b) and (c): let $z \in \mathbb{C}$ be a complex number with $|z| < 1$. Then the following infinite geometric series converges and has a simple closed form expression:

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}$$

Problem 6

Inverse systems can be used to compensate for imperfect measuring devices. Consider the case of a thermometer for measuring the temperature of a liquid. Because of the way the
thermometer works, it does not respond instantaneously to temperature changes. Let \( x(t) \) be the liquid temperature and \( y(t) \) be the output of the thermometer. Suppose we model the thermometer as an LTI system mapping \( x(t) \) to \( y(t) \). Suppose that when the liquid temperature over time is \( u(t) \) the thermometer reads
\[
s(t) = (1 - e^{-t/2})u(t)
\]

(a) Sketch \( s(t) \).
(b) Design a system that takes the output of the thermometer as input and produces an output equal to the instantaneous temperature of the liquid.

The impulse response of the thermometer system is the derivative of the step response:
\[
h(t) = \frac{1}{2}e^{-t/2}u(t)
\]
The inverse system has impulse response \( g(t) \) such that \( h(t) \ast g(t) = \delta(t) \). We can solve for the Fourier transform of \( g \) using \( H(w)G(w) = 1 \). This leads to \( g(t) = \delta(t) + 2\frac{d\delta(t)}{dt} \).