Problem 1

Suppose the Fourier series coefficients of a periodic, continuous-time signal with period 4 are the following:

\[ a_k = \begin{cases} 
  jk & |k| < 3 \\
  0 & \text{otherwise}
\end{cases} \]

Determine the signal \( x(t) \).

\[ x(t) = -2 \sin\left(\frac{\pi}{2} t\right) - 4 \sin(\pi t). \]

Problem 2

Consider the following three continuous-time signals with a fundamental period of \( T = 1/2 \):

\[ x(t) = \cos(4\pi t) \]
\[ y(t) = \sin(4\pi t) \]
\[ z(t) = x(t)y(t) \]

(a) Determine the Fourier series coefficients of \( x(t) \).

The nonzero Fourier series coefficients are \( a_1 = \frac{1}{2}, a_{-1} = \frac{1}{2} \).

(b) Determine the Fourier series coefficients of \( y(t) \).
The nonzero Fourier series coefficients are \( b_1 = \frac{1}{2j}, b_{-1} = \frac{-1}{2j} \).

(c) Use the results of parts (a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of \( z(t) = x(t)y(t) \).

The Fourier series coefficients of \( z(t) = x(t)y(t) \) are the following:

\[
 h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_1 b_{k-1} + a_{-1} b_{k+1}
\]

Therefore, the nonzero coefficients are \( h_2 = \frac{1}{4j}, h_{-2} = \frac{-1}{4j} \).

(d) Determine the Fourier series coefficients of \( z(t) \) through direct expansion of \( z(t) \) in trigonometric form, and compare you result with that of part (c).

\[
 \cos(4\pi t) \sin(4\pi t) = \frac{1}{2} [\sin(8\pi t) - \sin(0)] = \frac{1}{2} \cdot \frac{1}{2j} [e^{j2(4\pi t)} - e^{-j2(4\pi t)}]
\]

So the nonzero coefficients are \( h_2 = \frac{1}{4j}, h_{-2} = \frac{-1}{4j} \) as above.

**Problem 3**

Let \( S_A \) be the set of all complex-valued discrete signals, \( S_B \) be the set of complex-valued discrete signals that are bounded and \( S_S \) be the set of complex-valued discrete signals that are absolutely summable. The mathematical definitions are below.

\[
 S_A = \{ x : \mathbb{Z} \rightarrow \mathbb{C} \} \quad (1)
\]

\[
 S_B = \{ x : \mathbb{Z} \rightarrow \mathbb{C} \mid \exists b \in \mathbb{R} \text{ such that } |x[n]| \leq b \} \quad (2)
\]

\[
 S_S = \{ x : \mathbb{Z} \rightarrow \mathbb{C} \mid \exists b \in \mathbb{R} \text{ such that } \sum_{n=-\infty}^{\infty} |x[n]| \leq b \} \quad (3)
\]

(a) Show that \( S_S \subseteq S_B \subseteq S_A \).

If \( x \in S_S \) there is a \( b \) such that \( \sum |x[n]| \leq b \). Since each value in the sum is non-negative we must have that each value is at most \( b \). This implies \( x \in S_B \).

If \( x \in S_B \) then it is in \( S_A \) since \( S_A \) includes all signals with no restrictions.

(b) Show that if \( x_1, x_2 \in S_B \) then \( x_1 + x_2 \in S_B \) and \( x_1 - x_2 \in S_B \).
Since $x_1, x_2 \in S_B$ there exist $b_1$ and $b_2$ such that $|x_1[n]| \leq b_1$ and $|x_2[n]| \leq b_2$. Note that $|a + b| \leq |a| + |b|$ and $|a - b| \leq |a| + |b|$. Therefore $|x_1[n] + x_2[n]| \leq |x_1[n]| + |x_2[n]| \leq b_1 + b_2$ and $x_1 + x_2 \in S_B$ because each entry has absolute value at most $b = b_1 + b_2$. Similarly $|x_1[n] - x_2[n]| \leq |x_1[n]| + |x_2[n]| \leq b_1 + b_2$ and $x_1 - x_2 \in S_B$ because each entry has absolute value at most $b = b_1 + b_2$.

(c) Show that if $x_1, x_2 \in S_S$ then $x_1 + x_2 \in S_S$ and $x_1 - x_2 \in S_S$.

This is very similar to (b).

For each signal below, determine if it is bounded and/or absolutely summable.

(d) $x[n] = 2$

Bounded, not absolutely summable.

(e) $x[n] = \sin(n/10)$

Bounded, not absolutely summable.

(f) $x[n] = u(n)$

Bounded, not absolutely summable.

(g) $x[n] = u(n)\frac{1}{2^n}$

Bounded and absolutely summable.

(h) $x[n] = 1/n$ 

Not well defined at $n = 0$.

An arbitrary signal $x[n]$ that is zero outside of a finite range $n_0 \leq n \leq n_1$.

Bounded and absolutely summable.

**Problem 4**

A system is **stable** if it maps bounded signals to bounded signals.

(a) Let $h[n]$ be a signal that is zero everywhere except for $n \in A$ where $A$ is a finite set. Show that $C_h$ is stable.

Let $x$ be a bounded signal. Let $y[n] = h[n] * x[n]$. We need to show $y$ is bounded. Since
x is bounded, there is a b such that |x[n]| ≤ b. Let c = \max_{k \in A} |h[k]|.

\[ |y[n]| = \left| \sum_{k \in A} h[k]x[n - k] \right| \leq \sum_{k \in A} |h[k]| |x[n - k]| \leq \sum_{k \in A} cb \leq |A| cb \]

(b) Let \( h[n] = u(n) \). Is \( C_h \) stable? Justify your answer.

No. If the input is the bounded constant signal \( x[n] = 1 \) the output is not bounded.

(c) Let \( h[n] = u(n) \frac{1}{2^n} \). Is \( C_h \) stable? Justify your answer.

Yes. If the input \( x[n] \) is bounded there is a bound \( b \) such that |x[n]| ≤ b. Let \( y[n] \) be the output.

\[ |y[n]| = \left| \sum_{k=0}^{\infty} \frac{1}{2^k} x[n] \right| \leq \sum_{k=0}^{\infty} \frac{1}{2^k} |x[n]| \leq \sum_{k=0}^{\infty} \frac{1}{2^k} b = 2b \]

So the output \( y[n] \) is bounded by 2b.

**Problem 5**

Invertibility is relative...

We say a system is invertible over a set of signals \( S \) when it maps different signals from \( S \) to different signals. Equivalently, a system \( A \) is invertible over \( S \) if there exists another system \( B \) such that \( B(A(x)) = x \) for all signals \( x \in S \).

For each system below, determine if it is invertible over each of \( S_A \), \( S_B \), and \( S_S \).

(a) \( y[n] = 2x[n - 1] \)

Invertible over each set, with inverse \( z[n] = y[n + 1]/2 \).

(b) \( y[n] = x[2n] \) (subsampling)

Not invertible over any of the sets. Take \( x_1[n] = 0 \) and \( x_2[n] \) that is zero everywhere except \( x_2[1] = 1 \). Then \( x_1 \) and \( x_2 \) both map to the zero signal. Since both \( x_1 \) and \( x_2 \) are in \( S_S \), \( S_B \) and \( S_A \) the system is not invertible over any of the sets.

(c) \( y[n] = x[n] - x[n - 1] \) (derivative)

Not invertible over \( S_A \) or \( S_B \) because two different constant signals map to the same (zero) signal.
The derivative is invertible over $S_S$. To see this, suppose it was not. Then there must exist two different signals $x_1$ and $x_2$ in $S_S$ that map to the same signal. Since the system is linear $x_1 - x_2$ maps to the zero signal. Since $x_1 \neq x_2$ $x = x_1 - x_2$ is not zero everywhere. Part (c) of Problem 3 implies that $x \in S_S$. If $x$ is not zero everywhere there must be a $k$ for which $x[k] \neq 0$. But since $x$ maps to the zero signal we must have $x[k - 1] = x[k]$. Moreover $x[k - 2] = x[k - 1]$ and so on. Therefore $x$ is a constant non-zero signal. But such $x$ cannot be in $S_S$, which is a contradiction.

(d) $y[n] = x[n] + \frac{1}{2}x[n - 1000]$ (simple echo)

Not invertible over $S_A$ but is invertible over $S_B$ and $S_S$. The reasoning is similar to (c).