A Note on Graceful Graphs with Large Chromatic Numbers

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Abstract

A *graceful labeling* of a graph G with m edges is a function $f : V(G) \to \{0, \ldots, m\}$ such that distinct vertices receive distinct numbers and $\{|f(u) - f(v)| : uv \in E(G)\} = \{1, \ldots, m\}$. A graph is *graceful* if it has a graceful labeling. In [1] this question was posed: " Is there an n-chromatic graceful graph for $n \geq 6$?". In this paper it is shown that for any natural number n, there exists a graceful graph G with $\chi(G) = n$.

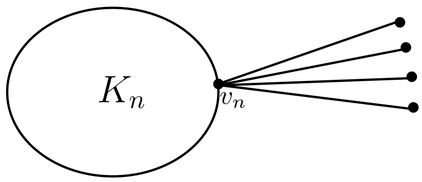
For a graph G we denote the set of vertices and the set of edges of G with $V(G)$ and $E(G)$, respectively. The chromatic number of a graph G, denoted by $\chi(G)$ is the minimum number of independent subsets into which $V(G)$ can be partitioned. A *graceful labeling* of a graph G with m edges is a function $f : V(G) \to \{0, \ldots, m\}$ such that distinct vertices receive distinct numbers and $\{|f(u)-f(v)| : uv \in E(G)\} = \{1, \ldots, m\}$. A graph is *graceful* if it has a graceful labeling. The label of an edge is the difference between the labels of its ends. In [2, p. 266] it has been conjectured that

Conjecture. Graceful graphs with arbitrary large chromatic number do not exist.

In the following theorem we give a negative answer to the above conjecture.

Theorem. *For any natural number n, there exists a graceful graph G such that $\chi(G) = n$.*

Proof. For $n = 1$ the assertion is trivial, thus suppose that $n \geq 2$. Let $v_1, \ldots, v_n$ be the vertices of the complete graph K_n. For each i, $1 \leq i \leq n$, consider 2^i as the label of v_i. First, we show that all edges of K_n in this labeling have different labels. Suppose $v_i v_j$ and $v_k v_l$ are two edges with the same labels. First assume that $i > j$ and $k > l$. Thus the labels of $v_i v_j$ and $v_k v_l$ are $2^i - 2^j$ and $2^k - 2^l$, respectively, and clearly they are distinct unless $i = k$ and $j = l$. This implies that two edges $v_i v_j$ and $v_k v_l$ are the same.
Now the greatest label of the vertices is 2^n and it is obvious that for each natural number n, we have $2^n > \frac{n(n-1)}{2}$. Add $2^n - \frac{n(n-1)}{2}$ new vertices to the complete graph K_n and join all of them to v_n. Let us call this graph by G. We claim that G has a graceful labeling. For each x, $x \in \{1, \ldots, 2^n\}$



which does not occur as a label of an edge in K_n label one of the new vertices with $2^n - x$. It is obvious that all edges have different labels. Moreover the labels of vertices are contained in $\{1, \ldots, 2^n\}$ and they are distinct. Indeed if the labels of two vertices are the same, then we conclude that the labels of two edges (of which one end is v_n) are the same, a contradiction. Note that n is an arbitrary number, and $\chi(G) = n$ $(n \geq 2)$. $\square$

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References

[1] G. Chartrand, H. Hevia, O. R. Oellermann, The Chromatic Number of a Factorization of a Graph, Bull. Inst. Combin. Appl. 20(1997), 33-56.

[2] G. Chartrand and L. Lesniak, Graphs & Digraphs, CHAPMAN & HALL/CRC, Fourth Edition, 2005.