Extending a Unit Vector to an Orthonormal Basis of 3-space

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Abstract

We show how to easily extend a unit vector to a right-handed orthonormal basis in 2-, 3-, and 4-space.

1 Introduction

Often in graphics, we have a unit vector, $\mathbf{u}$, that we wish to extend to a basis; for example, when you want to put a coordinate system (e.g. for texture-mapping) on a user-specified plane in 3-space, the natural specification of the plane is to give its normal, but this leaves the choice of plane-basis ambiguous up to a rotation in the plane. We describe the solution to this problem in 2, 3, and 4 dimensions.

2 2D and 4D

Oddly, 2D and 4D are the easy cases: to extend $\mathbf{u} = (x, y)$ to an orthonormal basis of $\Re^2$, let $\mathbf{v} = (-y, x)$. This corresponds to taking the complex number $x + iy$ and mutiplying by i, which rotates clockwise 90 degrees. To extend $\mathbf{u} = (a, b, c, d)$ to an orthonormal basis of $\Re^4$, let $\mathbf{v} = (-b, a, -d, c)$, $\mathbf{w} = (-c, d, a, -b)$ and $\mathbf{x} = (-d, -c, b, a)$. These corresponds to multiplying the quaternion $a + bi + cj + dk$ by i, j, and k, respectively.

3 3D

Oddly, 3D is harder – there's no continuous solution to the problem. If there were, we could take each unit vector $\mathbf{u}$ and extend it to a basis $\mathbf{u}, \mathbf{v}(\mathbf{u}), \mathbf{w}(\mathbf{u})$, where $\mathbf{v}$ is a continuous function. By drawing the vector $\mathbf{v}(\mathbf{u})$ at the tip of the vector $\mathbf{u}$, we'd create a continuous non-zero vector field on the sphere, which is impossible [1].

Here is a numerically stable and simple way to solve the problem, although it's not continuous in the input: Take the smallest entry (in absolute value) of $\mathbf{u}$ and set it to zero; swap the other two entries and negate the first of them.

The resulting vector $\bar{\mathbf{v}}$ is orthogonal to $\mathbf{u}$ and its length is at least $\sqrt{2/3} \approx .82$. Let $\mathbf{v} = \bar{\mathbf{v}}/||\bar{\mathbf{v}}||$, and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is an orthonormal basis. As a simple example, consider $\mathbf{u} = (-2/7, 6/7, 3/7)$. In this case, $\bar{\mathbf{v}} = (0, -3/7, 6/7)$, and hence $\mathbf{w} = \mathbf{u} \times \mathbf{v} = \frac{1}{7\sqrt{45}}(45, 12, 6)$.

Commentary: A more naive approach is to simply compute $\mathbf{v} = \mathbf{e_1} \times \mathbf{u}$ and $\mathbf{u} = \mathbf{v} \times \mathbf{w}$. This becomes ill-behaved when $\mathbf{u}$ and $\mathbf{e}_1$ are nearly parallel, at which point the naive approach substitutes $\mathbf{e_2}$ for $\mathbf{e_1}$. Our algorithm simply systematically avoids this problem. Another naive approach is to apply the Gram-Schmidt process to the set $\mathbf{u}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, discarding any vector whose projection onto the subspace orthogonal to the prior ones is shorter than, say, 1/10th. This, too, works, but uses multiple square-roots, and hence is worse computationally.

References

[1] Milnor, John, *Topology from the Differentiable Viewpoint*, University Press of Virginia, Charlottesville, 1965.