Lemma 1 (Safety) If \( \cdot \vdash e : JS \), then \( e \neq E[v["XMLHttpRequest"]], \) for any value \( v \).

Proof. By induction on the typing derivation \( \cdot \vdash e : JS \).

We only need to consider cases where \( e = E[e'] \). \( e \) is typable, there exist \( \Gamma, T \) such that \( \Gamma \vdash e' : T \).

We only need to consider cases where \( e' = e_1[e_2] \). The only typing rule for expressions of this form is T-GetField. By hypothesis of T-GetField, \( \Gamma \vdash e_2 : \text{NotXHR} \). By inversion, we conclude \( \Gamma \vdash e_2 : \text{NotXHR} \) by either T-Id or T-SafeValue. Consider each case:

- By T-Id, \( e_2 = x \), for some identifier \( x \). By definition of evaluation contexts, \( e_2 \) is a value, but identifiers are not values by definition. Hence, we have a contradiction.

- By the antecedent of T-SafeValue, \( e_2 \neq "XMLHttpRequest" \).

Lemma 2 (Subject Reduction) If \( \cdot \vdash e : JS \), and \( e \rightarrow e' \), then \( \cdot \vdash e' : JS \).

Proof. By induction on the typing derivation \( \cdot \vdash e : JS \) followed by case analysis on \( e \rightarrow e' \). The interesting cases are:

- T-IfSafe, which cannot occur, since the consequent is an open term.

- T-IfTrue-XHR, where:

\[
e = \text{if } ("XMLHttpRequest" === "XMLHttpRequest") \{ e_2 \} \text{ else } \{ e_3 \}
\]

in which the active expression is:

\[
e = E["XMLHttpRequest" === "XMLHttpRequest"]
\]

Evaluation proceeds by:

\[
"XMLHttpRequest" === "XMLHttpRequest" \rightarrow \text{true}
\]

\[
e \rightarrow E[\text{true}]
\]

\[1\]This inversion lemma needs to be proved by induction, due to subsumption.
\( e' = E[\text{true}] \)
\( e' = \text{if (true) \{ } e_2 \text{ \} else \{ } e_3 \text{ \}} \)

\( e' \) is typable by T-IfTrue, since \( \Gamma \vdash e_2 : \text{JS} \), by the hypothesis of T-IfTrue-XHR.

- **T-IfTrue**, where:

  \[
  e = \text{if (true) \{ } e_2 \text{ \} else \{ } e_3 \text{ \}} \\
  e = [\text{if (true) \{ } e_2 \text{ \} else \{ } e_3 \text{ \}}] \\
  \text{if (true) \{ } e_2 \text{ \} else \{ } e_3 \text{ \}} \leadsto e_2 \\
  e' = e_2
  \]

\( e_2 \) is typable by hypothesis of T-IfTrue.

Subject reduction for the remaining typing rules are conventional. We require a substitution lemma for evaluation of function applications and let-bindings. Since \( \lambda_{JS} \) is call-by-value, we can assume that in the lemma below, \( v \) is a value.

**Lemma 3 (Substitution)** If \( \Gamma, x : S \vdash e : T \) and \( \Gamma \vdash v : S \), then \( \Gamma \vdash e[x/v] : T \).

**Proof.** By induction on the typing derivation \( \Gamma, x : S \vdash e : T \).

The interesting case is is T-IfSafe, reproduced below:

\[
\frac{y \in \text{dom}(\Gamma) \quad \Gamma \vdash e_2 : \text{JS} \quad \Gamma[y : \text{NotXHR}] \vdash e_3 : \text{JS}}{\Gamma \vdash \text{if } y === "\text{XMLHttpRequest}" \text{then } e_2 \text{ else } e_3 : \text{JS}} \quad \text{(T-IfSafe)}
\]

Above, \( e = \text{if } y === "\text{XMLHttpRequest}" \text{then } e_2 \text{ else } e_3 \).

Our inductive hypotheses are:

1. If \( \Gamma, x : S \vdash e_2 : \text{JS} \), then \( \Gamma \vdash e_2[x/v] \).
2. If \( \Gamma[y/\text{NotXHR}], x : S \vdash e_3 : \text{JS} \), then \( \Gamma[y/\text{NotXHR}] \vdash e_3[x/v] : \text{JS} \).

We have two cases:

- **If** \( x \neq y \), then \( \Gamma \vdash e[x/v] : \text{JS} \) by T-IfSafe.

- **If** \( x = y \), then:

\[
e[x/v] = \text{if } (v === "\text{XMLHttpRequest}") \{ e_2[x/v] \} \text{ else } \{ e_3[x/v] \}
\]

We consider two subcases:

- \( v = "\text{XMLHttpRequest}" \). \( e[x/v] \) is typable by T-IfTRUE-XHR.
$- v \neq \ast XMLHttpRequest\ast$, so $v : \textbf{NotXHR}$ by $\text{T-SafeValue}$.

Since $\Gamma, x : S$ is an environment, $x \notin \text{dom}(\Gamma)$ by convention. Therefore, since $x = y$, in the second inductive hypothesis, $\Gamma[y/\textbf{NotXHR}] = \Gamma$.

Thus, we can rewrite the second inductive hypothesis as: If $\Gamma, x : S \vdash e_3 : \textbf{JS}$, then $\Gamma \vdash e_3[x/v] : \textbf{JS}$.

In addition, the third hypothesis of our instantiation of $\text{T-IfSafe}$ is simply $\Gamma \vdash e_2 : \textbf{JS}$.

Therefore, both inductive hypotheses apply and $\Gamma \vdash e[x/v] : \textbf{JS}$ by $\text{T-If}$. 