Topology and Distributed Computing

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SAN FRANCISCO, May 7 - Intel said on Friday that it was scrapping its development of two microprocessors, a move that is a shift in the company's business strategy....
Intel … [has] decided to focus its development efforts on “dual core” processors …. with two engines instead of one, allowing for greater efficiency because the processor workload is essentially shared.
The Future of Computing

• Speeding up uniprocessors is harder and harder
• Intel, Sun, AMD now focusing on “multi-core” architectures
• Soon, every computer will be a multiprocessor
What does this have to do with Theory?

Alan Mathison Turing. Inventor of the Turing Machine.
Turing Computability

- Mathematical model of computation
- What is (and is not) computable on uniprocessors
Time and Asynchrony

“Time is Nature’s way of making sure that everything doesn’t happen all at once.” (Anonymous, circa 1970)

Real world asynchrony: no clock in the sky
Asynchronous Computability?

- Mathematical model of asynchronous computation
- What is (and is not) asynchronously computable
Asynchrony Complicates Life

Seek help from modern mathematics!
FLP

• Fischer Lynch Paterson
• Showed that asynchronous computability ≠ Turing computability
• Consensus
  - Trivial in uniprocessor
  - Impossible with 1 asynchronous failure
• Reasoned directly about executions
Graph Theory

- Biran, Moran, Zachs 1988
- Single asynchronous failure
- Coordination problem
  - is a graph
- Problem is asynchronously computable
  - iff graph is connected
More Graph Theory

- Fischer Lynch 82, Dolev Strong 83, Merritt 85, Dwork Moses 90, ...
- Synchronous crash failures
- Computation state
  - is a graph
- Consensus not asynchronously computable
  - while graph is connected
Limitations of Graph Theory

• Asynchronous model
  - Multiple failures?
• Synchronous model
  - Problems beyond consensus?
• Need a more general notion of connectivity
Topological Approach

- Borowsky Gafni STOC 93
- Herlihy Shavit STOC 93, JACM 1999
- Saks Zaharoglou STOC 93, SIAM 2000
- Showed k-set agreement impossible
  - Generalization of consensus [Chaudhuri 90]
  - Open problem for several years
The Topological Approach

- **Computations as geometric object**
  - Use topological methods to show existence of “bad” execution.
- **Borowsky & Gafni**
  - Sperner’s Lemma
- **Herlihy & Shavit**
  - Simplicial complexes & homology
- **Saks & Zaharoglou**
  - Brouwer fixed-point theorem
Example: Autonomous Air Traffic Control

Pick your own altitude.
How many slots do we need to allow safe coordination?
Example Task: Renaming

• Process has input name (flight #)
• Must generate output name (altitude)
• Interested in comparison-based protocols:
  - Equality: A=B?
  - Order: A<B?
  - Nothing else (rules out trivial solutions)
History

• Proposed by Attiya, Bar-Noy, Dolev, Peleg, Reischuk

• They showed
  - Solution for $2n+1$ names
  - Impossibility for $n+2$ names
  - Intermediate values hot open question

• Long-standing open problem ...
  - Topological methods showed ...
  - Intermediate values also impossible
A Vertex

Point in high-dimensional Euclidean Space
Simplexes

- 0-simplex (vertex)
- 1-simplex (edge)
- 2-simplex (solid triangle)
- 3-simplex (solid tetrahedron)
Simplicial Complex
A Cycle

OK to think of it as oriented path

Also works in higher dimensions
A cycle is a boundary if it goes around a "solid" region.

Encompasses no "holes"
Not Every Cycle is a Boundary

Goes around a hole
Connectivity

• A complex is \( n \)-connected if
  - Every *cycle* of dimension \( n \) or less
  - Is also a *boundary*

• No “holes” in any dimension
  - Fundamental group is trivial
  - Higher homology groups trivial
Chromatic Complexes

Each $n$-simplex colored by $n$ distinct colors

Corresponding to process ids
Chromatic Simplicial Map

Color-preserving vertex-to-vertex map

That also carries simplexes to simplexes
Chromatic Simplicial Maps
Preserve Boundaries

A simplicial map can’t wrap a boundary around a “hole”
Vertex = Process State
Simplex = Global State
Complex = Global States
Decision Task

input complex (all possible flights)

Task spec relation

output complex (all consistent choices)
3 Planes, 3 Slots

input complex

output complex

input complex

output complex
3 Planes, 4 Slots
3 Planes, 4 Slots
3 Planes, 4 Slots
3 Planes, 4 Slots

input complex

output complex
Shared Memory

• Asynchronous
  - arbitrary delays
  - e.g., interrupts, page faults, etc.
• wait-free
  - Processes can fail or be slow
• Communication by reading and writing shared memory
Asynchronous Computability Theorem

A task has a wait-free protocol if and only if one can chromatically subdivide its input complex so that there exists a color preserving simplicial map to its output complex that refines the task spec.
Protocol Complex

- Each protocol defines a complex
  - vertex: sequence of messages received
  - simplex: compatible set of vertexes
- Induces subdivision of input complex
One-Round Protocol Complex

P runs solo

P and Q see one another

Q runs solo
One-Round Protocol Complex

R runs solo

(Some simplexes omitted for clarity)

P and Q run solo
Summary

Input complex

Protocol complex

Task def

Output complex

22-Jul-04
2 Planes 3 Slots

If view = (my_id, ⊥)
   decide 1
else if my_id < his_id
   decide 2
else decide 3

comparison based algorithm: only compares plane ids
3 Planes 4 Slots

- Impossibility by reduction
- Assume a protocol for 3 airplanes, 4 slots
- Choose
  - 0 if your name is even
  - 1 if your name is odd
- Result
  - Not all odd
  - Not all even
Reminder: Cannot Map Boundary Around a Hole

Hole is an obstruction
Output Complex (3 processes)
Protocol Complex (schematic)
Boundary = 2-Process Executions
Protocol Complex for One Process Execution

$P(\bigcirc)$ decides 1 \textit{WLOG}

$P(\bullet)$ decides 1 \textit{by symmetry}

$P(\bigcirc)$ decides 1 \textit{by symmetry}
2-Process execution might be mapped this way ...

Wraps around \(-1\) times

boundary

output
2-Process execution might be mapped that way ...

Wraps around +2 times
In General ...

Wraps around hole $3k-1 \neq 0$ times

QED!
Other Models of Computation

- Powerful atomic operations
  - Compare&swap, test&set, etc
- Synchronous fail-stop
  - Computation in rounds
3 Planes Need 5 Slots

AL991

DL227

AA082
Topological Approach to Asynchrony

Asynchrony

A multitude of concurrent executions

Reasoning about geometric objects

Topological Approach
The Glorious Future

• Our work
  - Asynchronous, wait-free, one-shot, RW memory

• Open problems
  - Long-lived computations
  - Other kinds of memory (compare&swap)
  - Randomization
  - Other progress conditions ...