14 Interpolation

14.1 Interpolation Basics

Goal: We would like to be able to define curves in a way that meets the following criteria:

1. Interaction should be natural and intuitive.
2. Smoothness should be controllable.
3. Analytic derivatives should exist and be easy to compute.
4. Representation should be compact.

**Interpolation** is when a curve passes through a set of “control points.”

![Figure 1: * Interpolation](image)

**Approximation** is when a curve approximates but doesn’t necessarily contain its control points.

![Figure 2: * Approximation](image)

**Extrapolation** is extending a curve beyond the domain of its control points.

**Continuity** - A curve is is $C^n$ when it is continuous in up to its $n^{\text{th}}$-order derivatives. For example, a curve is in $C^1$ if it is continuous and its first derivative is also continuous.

Consider a cubic interpolant — a 2D curve, $\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ where

\begin{align}
  x(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3, \\
  y(t) &= b_0 + b_1 t + b_2 t^2 + b_3 t^3,
\end{align}

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so

\[ x(t) = \sum_{i=0}^{3} a_i t^i = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{t}^T \vec{a}. \tag{3} \]

Here, \( \vec{t} \) is the basis and \( \vec{a} \) is the coefficient vector. Hence, \( \bar{c}(t) = \vec{t}^T \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \). (Note: \( T \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \) is a \( 4 \times 2 \) matrix).

There are eight unknowns, four \( a_i \) values and four \( b_i \) values. The constraints are the values of \( \bar{c}(t) \) at known values of \( t \).

**Example:**

For \( t \in (0, 1) \), suppose we know \( \bar{c}_j \equiv \bar{c}(t_j) \) for \( t_j = 0, \frac{1}{3}, \frac{2}{3}, 1 \) as \( j = 1, 2, 3, 4 \). That is,

\[
\begin{align*}
\bar{c}_1 &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \equiv \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}, \tag{4} \\
\bar{c}_2 &= \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \equiv \begin{bmatrix} x(1/3) \\ y(1/3) \end{bmatrix}, \tag{5} \\
\bar{c}_3 &= \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \equiv \begin{bmatrix} x(2/3) \\ y(2/3) \end{bmatrix}, \tag{6} \\
\bar{c}_4 &= \begin{bmatrix} x_4 \\ y_4 \end{bmatrix} \equiv \begin{bmatrix} x(1) \\ y(1) \end{bmatrix}. \tag{7}
\end{align*}
\]

So we have the following linear system,

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & (1/3)^2 & (1/3)^3 \\ 1 & 2/3 & (2/3)^2 & (2/3)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}, \tag{8}
\]

or more compactly, \( \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} = C \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \). Then, \( \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = C^{-1} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} \). From this we can find \( \vec{a} \) and \( \vec{b} \), to calculate the cubic curve that passes through the given points.
We can also place derivative constraints on interpolant curves. Let

\[ \vec{\tau}(t) = \frac{d\vec{c}(t)}{dt} = \frac{d}{dt} \left[ \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \end{array} \right] \left[ \begin{array}{c} \vec{a} \\ \vec{b} \end{array} \right] \]

that is, a different basis with the same coefficients.

**Example:**

Suppose we are given three points, \( t_j = 0, \frac{1}{2}, 1 \), and the derivative at a point, \( \vec{\tau}_2(\frac{1}{2}) \).

So we can write this as

\[ \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x'_2 & y'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/2 & (1/2)^2 & (1/2)^3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2(1/2) & 3(1/2)^2 \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}, \]

or

\[ \begin{bmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \vec{c}_3 \\ \vec{\tau}_2 \end{bmatrix} = C \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}, \]

which we can use to find \( \vec{a} \) and \( \vec{b} \):

\[ \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = C^{-1} \begin{bmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \vec{c}_3 \\ \vec{\tau}_2 \end{bmatrix}. \]

Unfortunately, polynomial interpolation yields unintuitive results when interpolating large numbers of control points; you can easily get curves that pass through the control points, but oscillate in very unexpected ways. Hence, direct polynomial interpolation is rarely used except in combination with other techniques.

### 14.2 Catmull-Rom Splines

**Catmull-Rom Splines** interpolate degree-3 curves with \( C^1 \) continuity and are made up of cubic curves.

A user specifies only the points \([\vec{p}_1, ... \vec{p}_N]\) for interpolation, and the tangent at each point is set to be parallel to the vector between adjacent points. So the tangent at \( \vec{p}_j \) is \( \kappa(\vec{p}_{j+1} - \vec{p}_{j-1}) \) for
endpoints, the tangent is instead parallel to the vector from the endpoint to its only neighbor). The
t value of $\kappa$ is set by the user, determining the “tension” of the curve.

Between two points, $\bar{p}_j$ and $\bar{p}_{j+1}$, we draw a cubic curve using $\bar{p}_j$, $\bar{p}_{j+1}$, and two auxiliary points on the tangents, $\kappa(\bar{p}_{j+1} - \bar{p}_{j-1})$ and $\kappa(\bar{p}_{j+2} - \bar{p}_j)$.

We want to find the coefficients $a_j$ when $x(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}^T \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}$, where the curve is defined as $\bar{c}(t) = \begin{bmatrix} c(t) & y(t) \end{bmatrix}$ (similarly for $g(t)$ and $b_j$). For the curve between $\bar{p}_j$ and $\bar{p}_{j+1}$, assume we know two end points, $\bar{c}(0)$ and $\bar{c}(1)$ and their tangents, $\bar{c}'(0)$ and $\bar{c}'(1)$.

To solve for $\bar{a}$, set up the linear system,

$$
\begin{bmatrix}
x(0) \\
x(1) \\
x'(0) \\
x'(1)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}.
$$

Then $\bar{x} = M\bar{a}$, so $\bar{a} = M^{-1}\bar{x}$. Substituting $\bar{a}$ in $x(t)$ yields

$$
x(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-3 & 3 & -2 & -1 \\
2 & -2 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_j \\
x_{j+1} \\
x_{j+1} - x_{j-1} \\
x_{j+2} - x_j
\end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}
\begin{bmatrix}
-\kappa & 0 & \kappa & 0 \\
2\kappa & \kappa - 3 & 3 - 2\kappa & -\kappa \\
-\kappa & 2 - \kappa & \kappa - 2 & \kappa
\end{bmatrix}
\begin{bmatrix}
x_{j-1} \\
x_j \\
x_{j+1} \\
x_{j+2}
\end{bmatrix}.
$$

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For the first tangent in the curve, we cannot use the above formula. Instead, we use:

\[ \vec{\tau}_1 = \kappa (\vec{p}_2 - \vec{p}_1) \]  

(21)

and, for the last tangent:

\[ \vec{\tau}_N = \kappa (\vec{p}_N - \vec{p}_{N-1}) \]  

(22)