The size bound follows immediately. The depth bound requires a more careful analysis.

Shown in Fig. 2.21(a) is a full adder together with notation showing the amount by which the length of a path from one input to another is increased in passing through it when the full-adder circuit used is that shown in Fig. 2.14 and described by Equation 2.6. From this it follows that

$$\begin{aligned} D_\Omega\left(c_{j+1}^{(l+1)}\right) &= \max\left(D_\Omega\left(c_j^{(l+1)}\right)+2, D_\Omega\left(s_j^{(l)}\right)+3\right) \\ D_\Omega\left(s_j^{(l+1)}\right) &= \max\left(D_\Omega\left(c_j^{(l+1)}\right)+1, D_\Omega\left(s_j^{(l)}\right)+2\right) \end{aligned}$$

for $2 \leq l$ and $0 \leq j \leq l-1$, where $s_{l-1}^{(l)} = c_{l-1}^{(l)}$. It can be shown by induction that $D_\Omega\left(c_j^{(k)}\right) = 2(k+j)-3$, $1 \leq j \leq k-1$, and $D_\Omega\left(s_j^{(k)}\right) = 2(k+j)-2$, $0 \leq j \leq k-2$, both for $2 \leq k$. (See Problem 2.16.) Thus, $D_\Omega\left(f_{\mathrm{count}}^{(n)}\right) = D_\Omega\left(c_{k-1}^{(k)}\right) = (4k-5)$. ■

We now use this bound to derive upper bounds on the size and depth of symmetric functions in the class $S_{n,m}$.

THEOREM 2.11.1 *Every symmetric function $f^{(n)} : \mathcal{B}^n \mapsto \mathcal{B}^m$ can be realized with the following circuit size and depth over the basis $\Omega = \{\wedge, \vee, \oplus\}$ where $\phi(k) = 5(2^k - k - 1)$:*

$$\begin{aligned} C_\Omega\left(f^{(n)}\right) &\leq m\lceil (n+1)/2\rceil + \phi(k) + 2(n+1) + (2\lceil \log_2(n+1)\rceil - 2)\sqrt{2(n+1)} \\ D_\Omega\left(f^{(n)}\right) &\leq 5\lceil \log_2(n+1)\rceil + \lceil \log_2\lceil \log_2(n+1)\rceil\rceil - 4 \end{aligned}$$

for $k = \lceil \log_2(n+1)\rceil$ even.

Proof Lemma 2.11.1 establishes bounds on the size and depth of the function $f_{\mathrm{count}}^{(n)}$ for $n = 2^k - 1$. For other values of n, let $k = \lceil \log_2(n+1)\rceil$ and fill out the $2^k - 1 - n$ variables with 0's.

The elementary symmetric functions are obtained by applying the value of $f_{\mathrm{count}}^{(n)}$ as argument to the decoder function. A circuit for this function has been constructed that has size $2(n+1) + (2\lceil \log_2(n+1)\rceil - 2)\sqrt{2(n+1)}$ and depth $\lceil \log_2\lceil \log_2(n+1)\rceil\rceil + 1$. (See Lemma 2.5.4. We use the fact that $2^{\lceil \log_2 m\rceil} \leq 2m$.) Thus, all elementary symmetric functions on n variables can be realized with the following circuit size and depth:

$$\begin{aligned} C_\Omega\left(e_0^{(n)}, e_1^{(n)}, \ldots, e_n^{(n)}\right) &\leq \phi(k) + 2(n+1) + (2\lceil \log_2(n+1)\rceil - 2)\sqrt{2(n+1)} \\ D_\Omega\left(e_0^{(n)}, e_1^{(n)}, \ldots, e_n^{(n)}\right) &\leq 4k - 5 + \lceil \log_2\lceil \log_2(n+1)\rceil\rceil + 1 \end{aligned}$$

The expansion of Equation (2.15) can be used to realize an arbitrary Boolean symmetric function. Clearly, at most n OR gates and depth $\lceil \log_2 n\rceil$ suffice to realize each one of m arbitrary Boolean symmetric functions. (Since the v_t are fixed, no ANDs are needed.) This number of ORs can be reduced to $(n-1)/2$ as follows: if $\lceil (n+1)/2\rceil$ or more elementary functions are needed, use the complementary set (of at most $\lfloor (n+1)/2\rfloor$ functions) and take the complement of the result. Thus, no more than $\lceil (n+1)/2\rceil - 1$ ORs are needed per symmetric function (plus possibly one NOT), and depth at most $\lceil \log_2\lfloor ((n+1)/2)\rfloor\rceil + 1$ $\leq \lceil \log_2(n+1)\rceil$. ■