Efficiently Building a Matrix to Rotate One Vector to Another

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Abstract. We describe an efficient (no square roots or trigonometric functions) method to construct the $3 \times 3$ matrix that rotates a unit vector $\mathbf{f}$ into another unit vector $\mathbf{t}$, rotating about the axis $\mathbf{f} \times \mathbf{t}$. We give experimental results showing this method is faster than previously known methods. An implementation in C is provided.

1. Introduction

Often in graphics, we have a unit vector, $\mathbf{f}$, that we wish to rotate to another unit vector, $\mathbf{t}$, by rotation in a plane containing both; in other words, we seek a rotation matrix $\mathbf{R}(\mathbf{f}, \mathbf{t})$ such that $\mathbf{R}(\mathbf{f}, \mathbf{t})\mathbf{f} = \mathbf{t}$. This paper describes a method to compute the matrix $\mathbf{R}(\mathbf{f}, \mathbf{t})$ from the coordinates of $\mathbf{f}$ and $\mathbf{t}$, without square root or trigonometric functions. Fast and robust C code can be found on the accompanying Web site.

2. Derivation

Rotation from $\mathbf{f}$ to $\mathbf{t}$ could be generated by letting $\mathbf{u} = \mathbf{f} \times \mathbf{t}/||\mathbf{f} \times \mathbf{t}||$, and then rotating about the unit vector $\mathbf{u}$ by $\theta = \arccos(\mathbf{f} \cdot \mathbf{t})$. A formula for the matrix that rotates about $\mathbf{u}$ by $\theta$ is given in Foley et al. [Foley et al. 90],

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namely
\[
\begin{pmatrix}
  u_x^2 + (1 - u_y^2) \cos \theta & u_x u_y (1 - \cos \theta) - y_z \sin \theta & u_x u_z + u_y \sin \theta \\
u_x u_y (1 - \cos \theta) + u_z \sin \theta & u_y^2 + (1 - u_x^2) \cos \theta & u_y u_z (1 - \cos \theta) - u_x \sin \theta \\
u_x u_z (1 - \cos \theta) - u_y \sin \theta & u_y u_z (1 - \cos \theta) + u_x \sin \theta & u_z^2 + (1 - u_y^2) \cos \theta
\end{pmatrix}
\]

The above involves \( \cos(\theta) \), which is just \( \mathbf{f} \cdot \mathbf{t} \), and \( \sin(\theta) \), which is \( ||\mathbf{f} \times \mathbf{t}|| \).

If we instead let
\[
\mathbf{v} = \mathbf{f} \times \mathbf{t} \\
c = \mathbf{f} \cdot \mathbf{t} \\
h = \frac{1-c}{1-c^2} = \frac{1-c}{\mathbf{v} \cdot \mathbf{v}}
\]

then, after considerable algebra, one can simplify the matrix to
\[
\mathbf{R}(\mathbf{f}, \mathbf{t}) = \begin{pmatrix}
c + hu_x^2 & hv_x v_y - v_z & hu_x v_z + v_y \\
hv_x v_y + v_z & c + hv_x^2 & hv_y v_z - v_x \\
hv_y v_z + v_x & hv_y v_z + v_x & c + hv_y^2
\end{pmatrix}
\] (1)

Note that this formula for \( \mathbf{R}(\mathbf{f}, \mathbf{t}) \) has no square roots or trigonometric functions.

When \( \mathbf{f} \) and \( \mathbf{t} \) are nearly parallel (i.e., \( ||\mathbf{f} \cdot \mathbf{t}|| > 0.99 \)), the computation of the plane that they define (and the normal to that plane, which will be the axis of rotation) is numerically unstable; this is reflected in our formula by the denominator of \( \mathbf{h} \) becoming close to zero.

In this case, we observe that a product of two reflections (angle-preserving transformations of determinant \(-1\)) is always a rotation, and that reflection matrices are easy to construct: For any vector \( \mathbf{u} \), the Householder matrix [Golub, Van Loan 96]

\[
\mathbf{H}(\mathbf{u}) = \mathbf{I} - \frac{2}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \mathbf{u}^t
\]

reflects the vector \( \mathbf{u} \) to \(-\mathbf{u}\), and leaves fixed all vectors orthogonal to \( \mathbf{u} \). In particular, if \( \mathbf{p} \) and \( \mathbf{q} \) are unit vectors, then \( \mathbf{H}(\mathbf{q} - \mathbf{p}) \) exchanges \( \mathbf{p} \) and \( \mathbf{q} \), leaving \( \mathbf{p} + \mathbf{q} \) fixed.

With this in mind, we choose a unit vector \( \mathbf{p} \) and build two reflection matrices: one that swaps \( \mathbf{f} \) and \( \mathbf{p} \), and the other that swaps \( \mathbf{t} \) and \( \mathbf{p} \). The product of these is a rotation that takes \( \mathbf{f} \) to \( \mathbf{t} \).

To choose \( \mathbf{p} \), we determine which coordinate axis \((x, y, \text{or} z)\) is most nearly orthogonal to \( \mathbf{f} \) (the one for which the corresponding coordinate of \( \mathbf{f} \) is smallest in absolute value) and let \( \mathbf{p} \) be a unit vector along that axis. We then build \( \mathbf{A} = \mathbf{H}(\mathbf{p} - \mathbf{f}) \), and \( \mathbf{B} = \mathbf{H}(\mathbf{p} - \mathbf{t}) \), and the rotation we want is \( \mathbf{R} = \mathbf{BA} \).
That is, if we let

\[
\mathbf{p} = \begin{cases} 
\hat{x}, & \text{if } |f_x| < |f_y| \text{ and } |f_z| < |f_x| \\
\hat{y}, & \text{if } |f_y| < |f_z| \text{ and } |f_x| < |f_z| \\
\hat{z}, & \text{if } |f_z| < |f_x| \text{ and } |f_y| < |f_x|
\end{cases}
\]

\[
\mathbf{u} = \mathbf{p} - \mathbf{f}
\]

\[
\mathbf{v} = \mathbf{p} - \mathbf{t},
\]

then the entries of \( \mathbf{R} \) are given by

\[
r_{ij} = \delta_{ij} - \frac{2}{\mathbf{u} \cdot \mathbf{u}} u_i u_j - \frac{2}{\mathbf{v} \cdot \mathbf{v}} v_i v_j + \frac{4\mathbf{u} \cdot \mathbf{v}}{(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v})} v_i u_j
\] (2)

where \( \delta_{ij} = 1 \) when \( i = j \) and \( \delta_{ij} = 0 \) when \( i \neq j \).

3. Performance

We tested the new method for performance against all previously known (by the authors) methods for rotating a unit vector into another unit vector. A naive way to rotate \( \mathbf{f} \) into \( \mathbf{t} \) is to use quaternions to build the rotation directly: Letting \( \mathbf{u} = \mathbf{v}/||\mathbf{v}|| \), where \( \mathbf{v} = \mathbf{f} \times \mathbf{t} \), and letting \( \phi = (1/2) \arccos(\mathbf{f} \cdot \mathbf{t}) \), we define \( \mathbf{q} = (\sin(\phi)\mathbf{u}; \cos \phi) \) and then convert the quaternion \( \mathbf{q} \) into a rotation via the method described by Shoemake [Shoemake 85]. This rotation takes \( \mathbf{f} \) to \( \mathbf{t} \), and we refer to this method as Naive. The second is called Cunningham and is just a change of bases [Cunningham 90]. Goldman [Goldman 90] gives a routine for rotating around an arbitrary axis: in our third method we simplified his matrix for our purposes; this method is denoted Goldman. All three of these require that some vector be normalized; the quaternion method requires normalization of \( \mathbf{v} \); the Cunningham method requires that one input be normalized, and then requires normalization of the cross-product. Goldman requires the normalized axis of rotation. Thus, the requirement of unit-vector input in our algorithm is not exceptional.

For the statistics below, we used 1,000 pairs of random normalized vectors \( \mathbf{f} \) and \( \mathbf{t} \). Each pair was fed to the matrix routines 10,000 times to produce accurate timings. Our timings were done on a Pentium II 400 MHz with compiler optimizations for speed on.

<table>
<thead>
<tr>
<th>Routine</th>
<th>Naive</th>
<th>Cunningham</th>
<th>Goldman</th>
<th>New Routine</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (s):</td>
<td>18.6</td>
<td>13.2</td>
<td>6.5</td>
<td>4.1</td>
</tr>
</tbody>
</table>

The fastest of previous known methods (Goldman) still takes about 50% more time than our new routine, and the naive implementation takes almost 350%
more time. Similar performance can be expected on most other architectures, since square roots and trigonometric functions are expensive to use.

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References


Web Information:

http://www.acm.org/jgt/papers/MollerHughes99

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