Lecture 24? Notes
24 April 2003

1 The orthogonal group

Last time we discussed Lie groups; this time I want to look at a couple of specific examples. For example, the set $O(n)$ of all $n \times n$ orthogonal matrices (matrices $B$ for which $B^tB = I$) turns out to be a Lie Group. The multiplication operation is matrix-multiply, inversion is matrix-inverse (which is the same as transpose for this special class of matrices). These are clearly smooth, being the restrictions of smooth maps. The only question is “Is $O(n)$ actually a manifold?”

Consider the map $g : M_{nn} \rightarrow M_{nn} : B \mapsto B^tB$.

This map is evidently smooth, and in an early homework, you computed its derivative and showed that it had rank $n(n + 1)/2$ by showing that every symmetric matrix was in its image. The implicit function theorem applies, and shows that $g^{-1}(I)$ is a smooth submanifold of $M_{nn}$ of dimension $n(n - 1)/2$.

The subgroup of $O(n)$ that consists of matrices whose determinant is $+1$ (instead of $-1$, the only other possibility) is called $SO(n)$, the “special” orthogonal group.

Let’s quickly look at the tangent space of $SO(n)$ at the identity element $I$. To do so, we need to find tangent vectors, which are (in one of our many definitions) just the derivatives of curves through the chosen point. So let

$$\gamma : (-1, 1) \rightarrow SO(n)$$

be a smooth curve with $\gamma(0) = I$. Because it’s a curve in $SO(n)$, we know that for each $s$,

$$\gamma(s)\gamma(s)^t = I.$$

Differentiating this tells us that

$$\gamma'(s)\gamma(s)^t + \gamma(s)\gamma'(s)^t = 0.$$

At $s = 0$, this becomes

$$\gamma'(0)\gamma(0)^t + \gamma(0)\gamma'(0)^t = 0,$$

i.e.

$$\gamma'(0)I + I\gamma'(0)^t = 0,$$

i.e.

$$\gamma'(0) + \gamma'(0)^t = 0.$$

which is to say that $\gamma'(0)$ is a skew-symmetric matrix. So every tangent vector to $SO(n)$ at $I$ is a skew-symmetric matrix. Since $T_I SO(n) \subset Y$, where $Y$ denotes
the $n \times n$ skew-symmetric matrices, and both are vector spaces of dimension $n(n - 1)/2$, they must in fact be equal.

Summary: The tangent space to $SO(n)$ at the identity is precisely the set of $n \times n$ skew-symmetric matrices. (It’s traditionally denoted $so(n)$, by the way.)

At some other point, $B$, of $SO(n)$ the tangent space is also relatively simple: $T_BSO(n)$ is, after all, just $(L_B)_*(T_1SO(n))$, i.e.,

$$T_BSO(n) = B \cdot so(n) = \{BR : R \in so(n)\}.$$  

That means that any tangent vector at $B$ is just $B$ times some skew-symmetric matrix. The same argument holds for right multiplication, so any tangent vector at $B$ is just some skew-symmetric matrix times $B$.

2 Application