**k-server**

In the $k$ server problem, given a graph (discrete metric space with associated distance $d(\cdot, \cdot)$) and $k$ servers placed on graph vertices, service a sequence of requests. Each request is a graph vertex and is served by bringing a server over to that node, at cost equal to the distance traveled by the server.

A state is a subset of $k$ of the $n$ graph vertices (possibly with repetitions) describing the positions of the servers. Let $A_0$ be the initial state and $r_t$ be the request at time $t$.

For any state $X$, let $w_t(X)$ denote the minimum cost of starting from $A_0$, serving requests $r_1, r_2, \ldots, r_t$, and ending in state $X$.

The working set algorithm, given the current state $A_{t-1}$, serves the next request $r_t$ by moving a server from $a \in A_{t-1}$ to $r_t$, changing the state into $A_t = A_{t-1} - a + r_t$, where $a$ is chosen so as to minimize $w_{t-1}(A_t) + d(a, r_t)$.

**Theorem 1** The working set algorithm for $k$-server is $(2k - 1)$-competitive.

**Proof:**

1. **Reduction to potential function analysis.** Up to an additive constant change in the total cost, we can always assume that our algorithm and OPT both start and end in the same state. Then we may use a telescoping sum and write the cost of OPT as $\sum_t (w_t(A_t) - w_{t-1}(A_{t-1}))$. Let the extended cost of serving request $t$ be: $d(a, r_t) + (w_t(A_t) - w_{t-1}(A_{t-1}))$. We will prove that the total extended cost is at most $2k$ times OPT, hence the theorem.

   We can always assume that the adversary picks a request sequence s.t. $r_t$ is never in $A_{t-1}$.

   To analyze the extended cost, first note that:

   **Lemma 1** $w_t(X) = \min_{x \in X} (w_{t-1}(X - x + r_t) + d(r_t, x))$

   Thus since $r_t \in A_t$, by Lemma 1 $w_t(A_t) = w_{t-1}(A_t)$. By the algorithm’s definition of $A_t$ and Lemma 1, $w_{t-1}(A_t) = w_t(A_{t-1}) - d(r_t, a)$. Substituting yields that the extended cost is:

   $$w_t(A_{t-1}) - w_{t-1}(A_{t-1}) \leq \max_X \{w_t(X) - w_{t-1}(X)\}.$$

   Note that the expression on the right hand side no longer depends on the algorithm but only on the work function.

   Given a work function $w$ and a vertex $a$, we say that a state $A$ is a minimizer with respect to $w, a$ if $A$ minimizes the expression $m_{w, a} = \min_A (w(A) - \sum_{x \in A} d(x, a))$. Given a work function $w$ we define a potential function,

   $$\Phi(w) = \min_U \{k w(U) + \sum_{u \in U} m_{w, a}\}.$$

   The crux of the proof is to argue that

   $$\max_X \{w_t(X) - w_{t-1}(X)\} \leq \Phi(w_t) - \Phi(w_{t-1}),$$

   hence the total extended cost is less than $\Phi(w_f) - \Phi(w_0)$, which is easily seen to be at most $2kw_f(A_f) + c$, hence the theorem.
2. Reduction to proving Lemmas 5 and 6. So, we now focus on $\Phi(w_t) - \Phi(w_{t-1})$. We easily observe:

**Lemma 2** $w_t(X) = \min_{x \in X} (w_t(X - x + r_t) + d(x, r_t))$.

from which, using the triangle inequality, with a short calculation we can infer:

**Lemma 3** We can assume that the $U$ minimizing $\Phi(w_t)$ is such that $r_t \in U$.

Let $U$ be that set, let $B_u$ be the minimizer for $w_t, u$ for each $u \in U, u \neq r_t$, and let $A$ be the minimizer for $w_t, r_t$. By definition of $\Phi(w_{t-1})$ we have

$$\Phi(w_{t-1}) \leq kw_{t-1}(U) + \sum_{u \in U, u \neq r_t} (w_{t-1}(B_u) - \sum_{b \in B} d(b, u)) + m_{w_{t-1}, r_t}.$$ 

Clearly,

**Lemma 4** The work function $w_t$ is monotone in $t$: for any $X$, $w_t(X) \geq w_{t-1}(X)$.

Applying this to $U$ and to the $B_u$’s, we deduce that

$$\Phi(w_t) - \Phi(w_{t-1}) \geq m_{w_t, r_t} - m_{w_{t-1}, r_t}.$$ 

The following lemma is a big step forward.

**Lemma 5** If $A$ is a minimizer for $w_{t-1}, r_t$ then $A$ is also a minimizer for $w_t, r_t$.

So we can take the same $A$ as a minimizer for both, and so

$$m_{w_t, r_t} - m_{w_{t-1}, r_t} = w_t(A) - w_{t-1}(A).$$

Finally, the other big step:

**Lemma 6** If $A$ is a minimizer with respect to $w_{t-1}, r_t$ then

$$w_t(A) - w_{t-1}(A) = \max_X \{w_t(X) - w_{t-1}(X)\}.$$ 

So we are done.

**Proving Lemmas 5 and 6.** Both proofs can be done by a few well-chosen algebraic manipulations relying on the following “quasi-convexity” property of the work function.

**Lemma 7** Fix $t$ and states $A$ and $B$. For any $a \in A$ there exists $b \in B$ such that

$$w_t(A) + w_t(B) \geq w_t(A - a + b) + w_t(A - b + a).$$

Lemma 7 is proved by induction over time and appealing to Lemma 1.