k-server

In the k server problem, given a graph (discrete metric space with associated distance $d(\cdot,\cdot)$) and k servers placed on graph vertices, service a sequence of requests. Each request is a graph vertex and is served by bring a server over to that node, at cost equal to the distance traveled by the server.

A state is a subset of k of the n graph vertices (possibly with repetitions) describing the positions of the servers. Let A_0 be the initial state and r_t be the request at time t.

For any state X, let $w_t(X)$ denote the minimum cost of starting from A_0, serving requests $r_1, r_2, \ldots, r_t$, and ending in state X.

The working set algorithm, given the current state A_{t-1}, serves the next request r_t by moving a server from $a \in A_{t-1}$ to r_t, changing the state into $A_t = A_{t-1} - a + r_t$, where a is chosen so as to minimize $w_{t-1}(A_t) + d(a, r_t)$.

Theorem 1 *The working set algorithm for k-server is* $(2k-1)$*-competitive.*

Proof:

1. Reduction to potential function analysis. Up to an additive constant change in the total cost, we can always assume that our algorithm and OPT both start and end in the same state. Then we may use a telescoping sum and write the cost of OPT as $\sum_t (w_t(A_t) - w_{t-1}(A_{t-1}))$. Let the extended cost of serving request t be: $d(a, r_t) + (w_t(A_t) - w_{t-1}(A_{t-1}))$. We will prove that the total extended cost is at most $2k$ times OPT. hence the theorem.

We can always assume that the adversary picks a request sequence s.t. r_t is never in A_{t_1}.

To analyze the extended cost, first note that:

Lemma 1 $w_t(X) = \min_{x \in X}(w_{t-1}(X - x + r_t) + d(r_t, x))$

Thus since $r_t \in A_t$, by Lemma 1 $w_t(A_t) = w_{t-1}(A_t)$. By the algorithm's definition of A_t and Lemma 1, $w_{t-1}(A_t) = w_t(A_{t-1}) - d(r_t, a)$. Substituting yields that the extended cost is:

$$w_t(A_{t-1}) - w_{t-1}(A_{t-1}) \leq \max_X \{w_t(X) - w_{t-1}(X)\}.$$

Note that the expression on the right hand side no longer depends on the algorithm but only on the work function.

Given a work function w and a vertex a, we say that a state A is a minimizer with respect to w, a if A minimizes the expression $m_{w,a} = \min_A (w(A) - \sum_{x \in A} d(x, a))$. Given a work function w we define a potential function,

$$\Phi(w) = \min_U \{kw(U) + \sum_{u \in U} m_{w,u}\}.$$

The crux of the proof is to argue that

$$\max_X \{w_t(X) - w_{t-1}(X)\} \leq \Phi(w_t) - \Phi(w_{t-1}),$$

hence the total extended cost is less than $\Phi(w_f) - \Phi(w_0)$, which is easily seen to be at most $2kw_f(A_f) + c$, hence the theorem.

2. Reduction to proving Lemmas 5 and 6. So, we now focus on $\Phi(w_t) - \Phi(w_{t-1})$. We easily observe:

Lemma 2 $w_t(X) = \min_{x \in X}(w_t(X - x + r_t) + d(x, r_t))$.

from which, using the triangle inequality, with a short calculation we can infer:

Lemma 3 *We can assume that the U minimizing $\Phi(w_t)$ is such that $r_t \in U$.*

Let U be that set, let B_u be the minimizer for w_t, u for each $u \in U, u \neq r_t$, and let A be the minimizer for w_t, r_t. By definition of $\Phi(w_{t-1})$ we have

$$\Phi(w_{t-1}) \leq k w_{t-1}(U) + \sum_{u \in U, u \neq r_t} (w_{t-1}(B_u) - \sum_{b \in B} d(b, u)) + m_{w_{t-1}, r_t}.$$

Clearly,

Lemma 4 *The work function w_t is monotone in t: for any X, $w_t(X) \geq w_{t-1}(X)$.*

Applying this to U and to the B_u's, we deduce that

$$\Phi(w_t) - \Phi(w_{t-1}) \geq m_{w_t, r_t} - m_{w_{t-1}, r_t}.$$

The following lemma is a big step forward.

Lemma 5 *If A is a minimizer for w_{t-1}, r_t then A is also a minimizer for w_t, r_t.*

So we can take the same A as a minimizer for both, and so

$$m_{w_t, r_t} - m_{w_{t-1}, r_t} = w_t(A) - w_{t-1}(A).$$

Finally, the other big step:

Lemma 6 *If A is a minimizer with respect to w_{t-1}, r_t then*

$$w_t(A) - w_{t-1}(A) = \max_X \{w_t(X) - w_{t-1}(X)\}.$$

So we are done.

Proving Lemmas 5 and 6. Both proofs can be done by a few well-chosen algebraic manipulations relying on the following "quasi-convexity" property of the work function.

Lemma 7 *Fix t and states A and B. For any $a \in A$ there exists $b \in B$ such that*

$$w_t(A) + w_t(B) \geq w_t(A - a + b) + w_t(A - b + a).$$

Lemma 7 is proved by induction over time and appealing to Lemma 1.