CSCI 1951-G – Optimization Methods in Finance
Part 07:
Portfolio Optimization

March 9–16, 2018
The portfolio optimization problem

How to best allocate our money to \( n \) risky assets \( S_1, \ldots, S_n \) with random returns?

- \( \mu_i \): expected return of asset \( i \) in a time interval;

<table>
<thead>
<tr>
<th>( \mu_i )</th>
<th>Stocks</th>
<th>Bonds</th>
<th>Money Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1073</td>
<td>0.0737</td>
<td>0.0627</td>
<td></td>
</tr>
</tbody>
</table>

- \( \Sigma \): variance-covariance \( n \times n \) matrix of returns, with:
  - \( \sigma_{ii} \): variance of the return of asset \( i \);
  - \( \sigma_{ij} \): covariance of the returns of assets \( i \) and \( j \).

<table>
<thead>
<tr>
<th>Covariance</th>
<th>Stocks</th>
<th>Bonds</th>
<th>MM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>0.02778</td>
<td>0.00387</td>
<td>0.00021</td>
</tr>
<tr>
<td>Bonds</td>
<td>0.00387</td>
<td>0.01112</td>
<td>-0.00020</td>
</tr>
<tr>
<td>MM</td>
<td>0.00021</td>
<td>-0.00020</td>
<td>0.00115</td>
</tr>
</tbody>
</table>
The portfolio optimization problem

Portfolio \( x = (x_1, \ldots, x_n) \), where
\( x_i \): proportion of money invested in asset \( i \).

Expected return: \( \mathbb{E}[x] = \mu_1 x_1 + \cdots + \mu_n x_n = \mu^T x \)

Variance: \( \text{Var}[x] = \sum_{i,j} \sigma_{ij} x_i x_j = x^T \Sigma x \)
\( \text{Var}[x] \geq 0 \), so \( \Sigma \) … is positive semidefinite
(we assume positive definite)

Feasible portfolios: set \( \mathcal{X} = \{ x : Ax = b, Cx \geq d \} \)
One constraint is
\[ \sum_{i=1}^{n} x_i = 1 \]
The portfolio optimization problem

*Efficient portfolio w.r.t. $R > 0$: the portfolio with *minimum variance* among all those with expected return at least $R$. (variants possible, e.g., )

*Markowitz’ mean-variance optimization:* find the efficient portfolio:

$$
\begin{align*}
\min & \quad x^T \Sigma x \\
\text{s.t.} & \quad \mu^T x \geq R \\
& \quad Ax = b \\
& \quad Cx \geq d
\end{align*}
$$

This optimization problem is … *convex.*
We assumed $\Sigma \geq 0$, so the optimal solution is … *unique.*
The portfolio optimization problem

\[ \begin{array}{c|ccc|ccc} \mu_i & \text{Stocks} & \text{Bonds} & \text{MM} \\
\hline \mu_i & 0.1073 & 0.0737 & 0.0627 \\
\end{array} \]

<table>
<thead>
<tr>
<th>Covariance</th>
<th>Stocks</th>
<th>Bonds</th>
<th>MM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>0.02778</td>
<td>0.00387</td>
<td>0.00021</td>
</tr>
<tr>
<td>Bonds</td>
<td>0.00387</td>
<td>0.01112</td>
<td>-0.00020</td>
</tr>
<tr>
<td>MM</td>
<td>0.00021</td>
<td>-0.00020</td>
<td>0.00115</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{min } & 0.02778 x_S^2 + 2 \cdot 0.00387 x_S x_B + 2 \cdot 0.00021 x_S x_M \\
& + 0.01112 x_B^2 - 2 \cdot 0.00020 x_B x_M + 0.00115 x_M^2 \\
\text{s.t. } & 0.1073 x_S + 0.0737 x_B + 0.0627 x_M \geq R \\
& x_S + x_B + x_M = 1 \\
& x_S \geq 0, x_B \geq 0, x_M \geq 0
\end{align*}
\]
The portfolio optimization problem

<table>
<thead>
<tr>
<th>Rate of Return $R$</th>
<th>Variance</th>
<th>Stocks</th>
<th>Bonds</th>
<th>MM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.065</td>
<td>0.0010</td>
<td>0.03</td>
<td>0.10</td>
<td>0.87</td>
</tr>
<tr>
<td>0.070</td>
<td>0.0014</td>
<td>0.13</td>
<td>0.12</td>
<td>0.75</td>
</tr>
<tr>
<td>0.075</td>
<td>0.0026</td>
<td>0.24</td>
<td>0.14</td>
<td>0.62</td>
</tr>
<tr>
<td>0.080</td>
<td>0.0044</td>
<td>0.35</td>
<td>0.16</td>
<td>0.49</td>
</tr>
<tr>
<td>0.085</td>
<td>0.0070</td>
<td>0.45</td>
<td>0.18</td>
<td>0.37</td>
</tr>
<tr>
<td>0.090</td>
<td>0.0102</td>
<td>0.56</td>
<td>0.20</td>
<td>0.24</td>
</tr>
<tr>
<td>0.095</td>
<td>0.0142</td>
<td>0.67</td>
<td>0.22</td>
<td>0.11</td>
</tr>
<tr>
<td>0.100</td>
<td>0.0189</td>
<td>0.78</td>
<td>0.22</td>
<td>0</td>
</tr>
<tr>
<td>0.105</td>
<td>0.0246</td>
<td>0.93</td>
<td>0.07</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8.1: Efficient Portfolios

Figure 8.1: Efficient Frontier and the Composition of Efficient Portfolios
The efficient frontier

\( R_{\text{min}}, R_{\text{max}} \): minimum and maximum expected returns for efficient portfolios.

\[
\sigma(R) : [R_{\text{min}}, R_{\text{max}}] \rightarrow \mathbb{R}, \sigma(R) = (x_R^T \Sigma x_R)^{1/2}
\]

where \( x_R \) is the efficient portfolio w.r.t. \( R \in [R_{\text{min}}, R_{\text{max}}] \).

The efficient frontier is the graph

\[
E = \{(R, \sigma(R)) : R \in [R_{\text{min}}, R_{\text{max}}]\}
\]
Maximizing the Sharpe ratio

Consider a *riskless asset* with *deterministic* return $r_f \leq R_{\text{min}}$ (why does it make sense?)
Consider *convex combinations* between a *risky* portfolio $x$ with the riskless asset

$$x_\theta = [(1 - \theta)x \quad \theta]^T$$

As $\theta$ varies, (for fixed $x$) the combinations form a line on the stdev/mean plot:

![Graph showing Capital Allocation Line (CAL) with different slopes for different risky portfolios](image)

Figure 8.4: Capital Allocation Line

For different choices of $x$, the *slope* of the line changes.
Maximizing the Sharpe ratio

Which Capital Allocation Line (CAL) is the best?

The CAL with the largest slope: the corresponding portfolio will have the lowest stdev for any given value of \( R \geq r_f \).
To which portfolio $x$ does the optimal CAL corresponds to?

The feasible $x$ that maximizes the slope:

$$h(x) = \frac{\mu^T x - r_f}{(x^T \Sigma x)^{1/2}}$$

The quantity $h(x)$ is known as the **Sharpe ratio** or the **reward-to-volatility ratio**.
Maximizing the Sharpe ratio

To find the *optimal risky portfolio* we solve

\[
\max \frac{\mu^T x - r_f}{(x^T \Sigma x)^{1/2}}
\]

s.t. \(Ax = b\)
\(Cx \geq d\)

The feasible region is *polyhedral*, but the objective function may be *non-concave*.

Let’s build an equivalent *convex quadratic program*. 
Maximizing the Sharpe ratio

\[ \mathcal{X} = \{ x : Ax = b, Cx \geq d \} \]
(includes full alloc. const., assumes \( \exists \hat{x} \in \mathcal{X} \text{ s.t. } \mu^T \hat{x} > r_f \))

\[ \mathcal{X}^+ = \{(x, k) : x \in \mathbb{R}^n, k \in \mathbb{R}^{++}, \frac{x}{k} \in \mathcal{X}\} \cup \{(0, 0)\} \]

The optimal risky portfolio is \( x^* = y^* / k^* \) where \( (y^*, k^*) \) is the optimal solution of:

\[
\begin{align*}
\min \ y^T \Sigma y \\
\text{s.t.} \ (y, k) \in \mathcal{X}^+ \\
(\mu - r_f \bar{1})^T y = 1
\end{align*}
\]

This is a quadratic convex program (why?)
You are a portfolio manager (!) and would like to understand how your portfolio manager “friend” Sally, the style of her portfolio i.e., the mix of stocks in it;

Sally is secretive on the mix, but publish the returns of her portfolio over time;

You also have access to the returns of index funds tracking different sectors of the market;

Definition (Return-Based Style Analysis (RBSA))

A technique using constrained optimization to determine the style of a portfolio using the return time series of the portfolio and of a number of other asset classes (factors).
Fundamentally a *linear model for regression*.

**Data**

- $R_t, t = 1, \ldots, T$: the return of Sally’s portfolio over $T$ fixed time intervals (e.g., $R_t$ is the monthly return, and $T = 12$ months);
- $F_{it}, i = 1, \ldots, n, t = 1, \ldots, T$: the returns of factor $i$ over $T$ fixed time intervals (same intervals as $R_t$);

**Model**

$$R_t = w_{1t}F_{1t} + w_{2t}F_{2t} + \cdots + w_{nt}F_{nt} + \varepsilon_t = F_t^Tw_t + \varepsilon_t$$

- $w_{it}$: *sensitivity* of $R_t$ to factor $i$;
- $\varepsilon_t$: *non-factor* return.
Interpretation

\[ R_t = w_{1t}F_{1t} + w_{2t}F_{2t} + \cdots + w_{nt}F_{nt} + \varepsilon_t = F_t^T w_t + \varepsilon_t \]

Assume the \( F_{it} \) are returns of passive investments (e.g., index funds); Then:

- \( F_t^T w_t \) is the return of a benchmark portfolio of passive investments;
- \( \varepsilon_t \) is the difference between the passive benchmark and the active strategy followed by Sally.

If the passive investments considered together are representative of the market, then \( \varepsilon_t \) measures the additional (or negative) return due to Sally’s ability as a portfolio manager.
Optimization problem

\[ R_t = w_{1t}F_{1t} + w_{2t}F_{2t} + \cdots + w_{nt}F_{nt} + \varepsilon_t = F_t^T w_t + \varepsilon_t \]

Additional assumptions/constraints:

- \( w_{it} = w_i \), i.e., the weights do not change over time.
- \( w_i > 0, \sum_{i=1}^{n} w_t = 1 \).

Constraints of our optimization problem:

\[
\begin{align*}
\text{min} & \quad ??? \\
\text{s.t.} & \quad \sum_{i=1}^{n} w_i = 1 \\
& \quad w_i \geq 0, \quad i = 1, \ldots, n
\end{align*}
\]

What about the objective function? Any idea?
Optimization problem (cont.)

• If $\varepsilon_t$ measures Sally’s *ability* as a portfolio manager, we can assume that it is *approximately constant over time*.

• I.e., we want the *plots of the returns* of Sally’s portfolios and of the benchmark portfolios to be *curves with approximately constant distance*.

• I.e., we want $\varepsilon_t$ to have the smallest possible *variance* over time.

**Formulation**

$$\min_{w \in \mathbb{R}^n} \text{Var}(\varepsilon_t) = \text{Var}(R_t - F_t^T w)$$

s.t. $\sum_{i=1}^{n} w_i = 1$

$$w_i \geq 0, i = 1, \ldots, n$$

The objective function is *convex*. 
Objective function

Let

\[ R = \begin{bmatrix} R_1 \\ \vdots \\ R_T \end{bmatrix}, \text{ and } F = \begin{bmatrix} F_1^T \\ \vdots \\ F_T^T \end{bmatrix}, \text{ and } e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \]

We have

\[
\text{Var}(R_t - F_t^T w) = \frac{1}{T} \sum_{i=1}^{T} (R_t - F_t^T w)^2 - \left( \frac{\sum_{t=1}^{T} (R_t - F_t^T w)}{T} \right)^2
\]

\[
= \frac{1}{T} \| R - Fw \|^2 - \left( \frac{e^T (R - Fw)}{T} \right)^2
\]

\[
= \frac{\| R \|^2 - 2R^T Fw + w^T F^T Fw}{T}
\]

\[
- \frac{(e^T R)^2 - 2e^T R - w^T F^T e e^T Fw}{T^2}
\]
Objective function

\[
\text{Var}(R_t - F_t^T w) = \frac{||R||^2 - 2R^T Fw + w^T F^T Fw}{T^2} - \frac{(e^T R)^2 - 2e^T R - w^T F^T e e^T Fw}{T^2}
\]

Reorganizing the terms as function of \( w \):

\[
\text{Var}(R_t - F_t^T w) = \left( \frac{||R||^2}{T} - \frac{(e^T R)^2}{T^2} \right) - 2 \left( \frac{R^T F}{T} - \frac{e^T R}{T^2} e^T F \right) w + w^T \left( \frac{1}{T} F^T F - \frac{1}{T} F^T e e^T F \right) w
\]

We have

\[
\frac{1}{T} F^T F - \frac{1}{T} F^T e e^T F = \frac{1}{T} F^T \left( I - \frac{e e^T}{T} \right) F
\]

The matrix \( M = I - e e^T / T \) is symmetric and positive semidefinite (eigenvalues: 0 and 1), and so is \( F^T M F \).

Hence the objective function is convex.